

ANISOTROPIC HARDY-LORENTZ SPACES WITH VARIABLE EXPONENTS

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ABSTRACT. In this paper we introduce Hardy-Lorentz spaces with variable exponents associated to dilations in \mathbb{R}^n . We establish maximal characterizations and atomic decompositions for our variable exponent anisotropic Hardy-Lorentz spaces.

1. INTRODUCTION

The celebrated Fefferman and Stein's paper [24] has been crucial in the development of the real variable theory of Hardy spaces. In [24] the tempered distributions in the Hardy spaces $H^p(\mathbb{R}^n)$ were characterized as those ones such that certain maximal functions are in $L^p(\mathbb{R}^n)$. Coifman [9] and Latter [34] obtained atomic decompositions of the elements of the Hardy spaces $H^p(\mathbb{R}^n)$. Here, $0 < p < \infty$ and $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ provided that $1 < p < \infty$.

Many authors have investigated Hardy spaces in several settings. Some generalizations substitute the underlying domain \mathbb{R}^n by other ones (see, for instance, [6], [8], [11], [39], [44] and [48]). Also, Hardy spaces associated with operators have been defined (see [20], [21], [30], [31] and [49], amongst others). If X is a function space, the Hardy space $H(\mathbb{R}^n, X)$ on \mathbb{R}^n modeled on X consists of all those tempered distributions f on \mathbb{R}^n such that the maximal function $\mathcal{M}(f)$ of f is in X . The definition of the maximal operator \mathcal{M} will be shown below. The classical Hardy space $H^p(\mathbb{R}^n)$ is the Hardy space on \mathbb{R}^n modeled on $L^p(\mathbb{R}^n)$. If ν is a weight on \mathbb{R}^n and $L^p(\mathbb{R}^n, \nu)$ denotes the weighted Lebesgue space, the Hardy space $H(\mathbb{R}^n, L^p(\mathbb{R}^n, \nu))$ was investigated in [26]. The Hardy space $H(\mathbb{R}^n, L^{p,q}(\mathbb{R}^n))$ where $L^{p,q}(\mathbb{R}^n)$ represents the Lorentz space has been studied in [1], [23], [25], [28] and [29]. The Hardy space $H(\mathbb{R}^n, \Lambda^p(\phi))$ on \mathbb{R}^n modeled on generalized Lorentz space $\Lambda^p(\phi)$ was studied by Almeida and Caetano [2]. The variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, investigated by [14], [40], [43] and [50], is the space $H(\mathbb{R}^n, L^{p(\cdot)}(\mathbb{R}^n))$ on \mathbb{R}^n modeled on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

By $S(\mathbb{R}^n)$, as usual, we denote the Schwartz function class on \mathbb{R}^n and by $S'(\mathbb{R}^n)$ its dual space. If $\varphi \in S(\mathbb{R}^n)$, the radial maximal function $\mathcal{M} = M_\varphi$ used to characterized Hardy spaces is defined by

$$\mathcal{M}(f) = \sup_{t>0} |f * \varphi_t|, \quad f \in S'(\mathbb{R}^n),$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$, $x \in \mathbb{R}^n$ and $t > 0$. Bownik [4] studied anisotropic Hardy spaces on \mathbb{R}^n associated with dilations in \mathbb{R}^n . If A is a dilation (the definition will be specified later) in \mathbb{R}^n , for every $k \in \mathbb{Z}$, we define

$$\varphi_{A,k}(x) = |\det A|^{-k} \varphi(A^{-k}x), \quad x \in \mathbb{R}^n,$$

and the maximal function $\mathcal{M}_A = M_{A,\varphi}$ associated with A is given by

$$\mathcal{M}_A(f) = \sup_{k \in \mathbb{Z}} |f * \varphi_{A,k}|, \quad f \in S'(\mathbb{R}^n).$$

Bownik [4] characterizes anisotropic Hardy spaces by maximal functions like \mathcal{M}_A . Recently, Liu, Yang, and Yuan [35] have extended Bownik's results ([4]) by studying anisotropic Hardy spaces on \mathbb{R}^n modeled on Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

Ephremidze, Kokilashvili and Samko [22] introduced variable exponent Lorentz Spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. In this paper we define anisotropic Hardy spaces on \mathbb{R}^n associated with a dilation A modelled on $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. These Hardy spaces are represented by $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and they are called variable exponent anisotropic Hardy-Lorentz spaces on \mathbb{R}^n . We characterize the tempered distributions in

Date: Saturday 6th August, 2016.

2010 *Mathematics Subject Classification.* 42B30 (42B25, 42B35).

Key words and phrases. Variable exponent Hardy spaces, Hardy-Lorentz spaces, anisotropic Hardy spaces, maximal functions, atomic decomposition.

The authors are partially supported by MTM2013-44357-P.

$H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ by using anisotropic maximal function \mathcal{M}_A . Also, we obtain atomic decompositions for the elements of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Our results extend those ones in [35] to variable exponent setting.

Before establishing the results of this paper we recall the definitions and properties about anisotropy and variable exponent Lebesgue and Lorentz spaces we will need.

An exhaustive and systematic study about variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ can be found in the monograph [18]. Here $p : \Omega \rightarrow (0, \infty)$ is a measurable function. We assume that $0 < p_-(\Omega) \leq p_+(\Omega) < \infty$, where $p_-(\Omega) = \inf_{x \in \Omega} p(x)$ and $p_+(\Omega) = \sup_{x \in \Omega} p(x)$. The space $L^{p(\cdot)}(\Omega)$ is the collection of all measurable functions f such that, for some $\lambda > 0$, $\varepsilon(f/\lambda) < \infty$, where

$$\varepsilon(f) = \int_{\Omega} |f(x)|^{p(x)} dx.$$

We define $\| \cdot \|_{p(\cdot)}$ as follows

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}, \quad f \in L^{p(\cdot)}(\Omega).$$

If $p_-(\Omega) \geq 1$, then $\| \cdot \|_{p(\cdot)}$ is a norm and $(L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)})$ is a Banach space. However, if $p_-(\Omega) < 1$, then $\| \cdot \|_{p(\cdot)}$ is a quasinorm and $(L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)})$ is a quasi Banach space.

A crucial problem concerning to variable exponent Lebesgue spaces is to describe the exponents p for which the Hardy-Littlewood maximal function is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ (see, [12], [17], [41], and [42], amongst others). As it is shown in [13], [15] and [27], the boundedness of the Hardy-Littlewood maximal function together with extensions of Rubio de Francia's extrapolation theorem lead to prove the boundedness in variable exponent Lebesgue spaces of a wide class of operators and vector valued inequalities in $L^{p(\cdot)}(\mathbb{R}^n)$ for the Hardy-Littlewood maximal operator. These ideas also work in the variable exponent Lorentz spaces, introduced by Ephremidze, Kokilashvili and Samko [22] and they will play a fundamental role in the proof of some of our main results.

The Lorentz spaces were introduced in [36] and [37] as a generalization of classical Lebesgue spaces. The theory of Lorentz spaces can be encountered in [3] and [7]. Assume that f is a measurable function. We define the distribution function $\mu_f : [0, \infty) \rightarrow [0, \infty]$ associated with f by

$$\mu_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad s \in [0, \infty).$$

Here, $|E|$ denotes the Lebesgue measure of E , for every Lebesgue measurable set E . The non-increasing equimeasurable rearrangement $f^* : [0, \infty) \rightarrow [0, \infty]$ of f is defined by

$$f^*(t) = \inf\{s \geq 0 : \mu_f(s) \leq t\}, \quad t \in [0, \infty).$$

If $0 < p, q < \infty$, the measurable function f is in the Lorentz space $L^{p,q}(\mathbb{R}^n)$ provided that

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \left(\int_0^\infty t^{q/p-1} (f^*(t))^q dt \right)^{1/q} < \infty.$$

$L^{p,q}(\mathbb{R}^n)$ is complete and it is normable for $1 < p < \infty$ and $1 \leq q < \infty$.

Variable exponent Lorentz spaces have been defined in two different ways: one of them by Ephremidze, Kokilashvili and Samko [22] and the other one by Kempka and Vybíral [32]. In this paper we consider the space defined in [22].

For every $a \geq 0$ we denote by \mathfrak{P}_a the set of measurable functions $p : (0, \infty) \rightarrow (0, \infty)$ such that $a < p_-(0, \infty) \leq p_+(0, \infty) < \infty$. By \mathbb{P} we represent the class of bounded measurable functions $p : (0, \infty) \rightarrow (0, \infty)$ such that there exist the limits

$$p(0) =: \lim_{t \rightarrow 0^+} p(t) \quad \text{and} \quad p(\infty) =: \lim_{t \rightarrow +\infty} p(t)$$

and the following conditions are satisfied

$$|p(t) - p(0)| \leq \frac{C}{|\ln t|}, \quad \text{for } 0 < t \leq 1/2$$

and

$$|p(t) - p(\infty)| \leq \frac{C}{\ln(e+t)}, \quad \text{for } t \in (0, \infty).$$

We also denote $\mathbb{P}_a = \mathbb{P} \cap \mathfrak{P}_a$, for every $a \geq 0$.

Let $p, q \in \mathfrak{P}_0$. We represent by $(p(\cdot), q(\cdot))$ -Lorentz space $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ the space of all those measurable functions f on \mathbb{R}^n such that $t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \in L^{q(\cdot)}(0, \infty)$. We define

$$\|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)} = \|t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t)\|_{L^{q(\cdot)}(0, \infty)}, \quad f \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n).$$

We also consider the average f^{**} of f^* given by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t \in (0, \infty),$$

and

$$\|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{(1)} = \|t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^{**}(t)\|_{L^{q(\cdot)}(0, \infty)}, \quad f \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n).$$

$\|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{(1)}$ satisfies the triangular inequality provided that $q_-((0, \infty)) \geq 1$. It is clear that $\|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{(1)}$. According to [22, Theorem 2.4] if $p \in \mathbb{P}_0$, $q \in \mathbb{P}_1$, $p(0) > 1$, and $p(\infty) > 1$, there exists $C > 0$ for which

$$\|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{(1)} \leq C \|f\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}, \quad f \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n).$$

If $p, q \in \mathbb{P}_1$, $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ is a Banach function space (in the sense of [3]) and the dual space $(\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n))'$ coincides with $\mathcal{L}^{p'(\cdot), q'(\cdot)}(\mathbb{R}^n)$ [22, Lemma 2.7 and Theorem 2.8]. Here as usual if $r : (0, \infty) \rightarrow (1, \infty)$, $r' = \frac{r}{r-1}$. The behaviour of the anisotropic Hardy-Littlewood maximal function on $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ will be very useful in the sequel. According to [22, Theorem 3.12], the classical Hardy-Littlewood maximal operator is bounded from $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ into itself provided that $p, q \in \mathbb{P}_1$.

The main definitions and properties about the anisotropic setting we will use in this paper can be found in [4].

Suppose that A is an expansive dilation matrix in \mathbb{R}^n , that is, a $n \times n$ real matrix such that $\min_{\lambda \in \sigma(A)} |\lambda| > 1$ where $\sigma(A)$ represents the set of eigenvalues of A . We say that a measurable function $\rho : \mathbb{R}^n \rightarrow [0, \infty)$ is a homogeneous quasinorm associated with A when the following properties hold:

- (a) $\rho(x) = 0$ if, and only if, $x = 0$;
- (b) $\rho(Ax) = |\det A| \rho(x)$, $x \in \mathbb{R}^n$;
- (c) $\rho(x + y) \leq H(\rho(x) + \rho(y))$, $x, y \in \mathbb{R}^n$, for certain $H \geq 1$.

If P is a nondegenerate $n \times n$ matrix, the set Δ defined by

$$\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$$

is called the ellipsoid generated by P . According to [4, Lemma 2.2, p. 5] there exists an ellipsoid Δ with Lebesgue measure 1 and such that, for certain $r_0 > 1$,

$$\Delta \subseteq r_0 \Delta \subseteq A\Delta.$$

From now on, the ellipsoid Δ satisfying the above properties is fixed. For every $k \in \mathbb{Z}$, we define $B_k = A^k \Delta$ and denote by ω the smallest integer such that $2B_0 \subset B_\omega$. We have that, for every $k \in \mathbb{Z}$, $|B_k| = b^k$, where $b = |\det A|$, and $B_k \subset r_0 B_k \subset B_{k+1}$.

The step quasinorm ρ_A on \mathbb{R}^n is defined by

$$\rho_A(x) = \begin{cases} b^k, & x \in B_{k+1} \setminus B_k, \quad k \in \mathbb{Z}, \\ 0, & x = 0. \end{cases}$$

Thus, ρ_A is a homogeneous quasinorm associated with A .

By [4, Lemma 2.4, p. 6] if ρ is any quasinorm associated with A , then ρ_A and ρ are equivalent, that is, for a certain $C > 0$,

$$\rho(x)/C \leq \rho_A(x) \leq C\rho(x), \quad x \in \mathbb{R}^n.$$

The triplet $(\mathbb{R}^n, \rho_A, |\cdot|)$, where $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^n , is a space of homogeneous type in the sense of Coifman and Weiss [10].

We now define maximal functions in our anisotropic setting. Suppose that $\varphi \in S(\mathbb{R}^n)$ and $f \in S'(\mathbb{R}^n)$. The radial maximal function $M_\varphi^0(f)$ of f with respect to φ is defined by

$$M_\varphi^0(f)(x) = \sup_{k \in \mathbb{Z}} |(f * \varphi_k)(x)|,$$

where $\varphi_k(x) = b^{-k} \varphi(A^{-k}x)$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Since the matrix A is fixed we do not refer it in the notation of maximal functions.

The nontangential maximal function $M_\varphi(f)$ with respect to φ is given by

$$M_\varphi(f)(x) = \sup_{k \in \mathbb{Z}, y \in x+B_k} |(f * \varphi_k)(y)|, \quad x \in \mathbb{R}^n.$$

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $s(\alpha) = \alpha_1 + \dots + \alpha_n$. Let $N \in \mathbb{N}$. We consider the set

$$S_N = \{\varphi \in S(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \rho_A(x)^N |D^\alpha \varphi(x)| \leq 1, \quad \alpha \in \mathbb{N}^n \text{ and } s(\alpha) \leq N\}.$$

Here $D^\alpha = \frac{\partial^{s(\alpha)}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, when $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

The radial grandmaximal function $M_N^0(f)$ of f of order N is defined by

$$M_N^0(f) = \sup_{\varphi \in S_N} M_\varphi^0(f).$$

The nontangential grandmaximal function $M_N(f)$ of f of order N is given by

$$M_N(f) = \sup_{\varphi \in S_N} M_\varphi(f).$$

We now define variable exponent anisotropic Hardy-Lorentz spaces. Let $N \in \mathbb{N}$ and $p, q \in \mathfrak{P}_0$. The $(p(\cdot), q(\cdot))$ -anisotropic Hardy-Lorentz space $H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ associated with A is the set of all those $f \in S'(\mathbb{R}^n)$ such that $M_N(f) \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$. On $H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ we consider the quasinorm $\| \cdot \|_{H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}$ defined by

$$\|f\|_{H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)} = \|M_N(f)\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}, \quad f \in H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A).$$

Our first result establishes that the space $H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ actually does not depend on N provided that N is large enough. Furthermore, we prove that $H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ can be characterized also by using the maximal functions M_φ^0 , M_φ and M_N^0 .

Theorem 1.1. *Let $f \in S'(\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^n)$ such that $\int \varphi \neq 0$. Assume that $p, q \in \mathbb{P}_0$. Then, the following assertions are equivalent.*

- (i) *There exists $N_0 \in \mathbb{N}$ such that, for every $N \geq N_0$, $f \in H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.*
- (ii) *$M_\varphi(f) \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$.*
- (iii) *$M_\varphi^0(f) \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$.*

Moreover, for every $g \in S'(\mathbb{R}^n)$ the quantities $\|M_N(g)\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}$, $N \geq N_0$, $\|M_\varphi^0(g)\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}$ and $\|M_\varphi(g)\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}$ are equivalent.

According to Theorem 1.1 we write $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ to refer $H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$, for every $N \geq N_0$.

In order to prove this theorem we follow the ideas developed by Bownik [4, §7] (see also [35, §4]) but we need to make some modifications due to that decreasing rearrangement and variable exponents appear.

Let $1 < r \leq \infty$, $s \in \mathbb{N}$ and $p, q \in \mathfrak{P}_0$. We say that a measurable function a on \mathbb{R}^n is a $(p(\cdot), q(\cdot), r, s)$ -atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ when a satisfies

- (a) $\text{supp } a \subseteq x_0 + B_k$.
- (b) $\|a\|_r \leq b^{k/r} \|\chi_{x_0+B_k}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1}$.
- (c) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$, for every $\alpha \in \mathbb{N}^n$ such that $s(\alpha) \leq s$.

Here, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

In the next result we characterize the distributions in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ by atomic decompositions.

Theorem 1.2. *Let $p, q \in \mathbb{P}_0$.*

- (i) *There exist $s_0 \in \mathbb{N}$ and $C > 0$ such that if, for every $j \in \mathbb{N}$, $\lambda_j > 0$ and a_j is a $(p(\cdot), q(\cdot), \infty, s_0)$ -atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, satisfying that*

$$\sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n), \text{ then } f = \sum_{j \in \mathbb{N}} \lambda_j a_j \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$$

and

$$\|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)} \leq C \left\| \sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}.$$

If also $p(0) \leq q(0)$, then there exists $r_0 > 1$ such that for every $r_0 < r < \infty$ the above assertion is true when $(p(\cdot), q(\cdot), \infty, s_0)$ -atoms are replaced by $(p(\cdot), q(\cdot), r, s_0)$ -atoms.

- (ii) There exists $s_0 \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s \geq s_0$, and $1 < r \leq \infty$, we can find $C > 0$ such that, for every $f \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$ and a $(p(\cdot), q(\cdot), r, s)$ -atom a_j associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, satisfying that $\sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j + B_{\ell_j}}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{x_j + B_{\ell_j}} \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$, $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$ and

$$\left\| \sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j + B_{\ell_j}}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{x_j + B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}.$$

From Theorem 1.2 it follows that, under its hypothesis, for certain $C > 0$ and for every $f \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$

$$\frac{1}{C} \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)} \leq \inf \left\| \sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j + B_{\ell_j}}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{x_j + B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)},$$

where the infimum is taken over all the sequences $(\lambda_j)_{j=1}^\infty \subset (0, \infty)$ and $(a_j)_{j=1}^\infty$ of $(p(\cdot), q(\cdot), r, s)$ -atoms satisfying that

$$\sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j + B_{\ell_j}}\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{x_j + B_{\ell_j}} \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$$

and $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$, being a_j associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, for every $j \in \mathbb{N}$.

In our proof of Theorem 1.2 a vector valued inequality, involving the Hardy-Littlewood maximal function in our anisotropic setting, plays an important role. The mentioned maximal function is defined by

$$M_{HL}(f)(x) = \sup_{k \in \mathbb{Z}, y \in x + B_k} \frac{1}{b^k} \int_{y + B_k} |f(z)| dz, \quad x \in \mathbb{R}^n.$$

After proving a version of [22, Theorem 3.12] for M_{HL} , by using an extension of Rubio de Francia extrapolation Theorem (see [13], [15], and [27]), we can establish the following result.

Proposition 1.3. *Assume that $p, q \in \mathbb{P}_1$. For every $r \in (1, \infty)$ there exists $C > 0$ such that*

$$\left\| \left(\sum_{j \in \mathbb{N}} M_{HL}(f_j)^r \right)^{1/r} \right\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^r \right)^{1/r} \right\|_{\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)},$$

for each sequence $(f_j)_{j \in \mathbb{N}}$ of functions in $L_{loc}^1(\mathbb{R}^n)$.

Remark 1.1. *We do not know if the the last vectorial inequality holds when the Lorentz space $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ is replaced by the variable exponent Lorentz space $L_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ introduced by Kempka and Vybíral [32]. In order to apply extrapolation technique it is necessary to know the associated Köthe dual space $(L_{p(\cdot), q(\cdot)}(\mathbb{R}^n))^*$ of $L_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$, but its characterization is, as far we know, an open question.*

Also in order to prove Theorem 1.2 we need to establish that $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A) \cap L_{loc}^1(\mathbb{R}^n)$ is a dense subspace of $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. At this point a careful study of Calderón-Zygmund decomposition of the distributions in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ must be made.

Ours Theorems 1.1 and 1.2, so far we know, represent new results even in the isotropic (classical) case.

The paper is organized as follows. A proof of Theorem 1.1 is presented in Section 2 where, besides, we prove the main properties of variable exponent anisotropic Hardy-Lorentz spaces. Next, in Section 3, Calderón-Zygmund decompositions in our setting are investigated. The proof of Theorem 1.2, which is presented distinguishing the cases $r = \infty$ and $r < \infty$, is included in Section 4.

Throughout this paper C always denotes a positive constant that can change its value from a line to another one.

2. MAXIMAL CHARACTERIZATIONS (PROOF OF THEOREM 1.1)

From now on, for simplicity, we write $\|\cdot\|_{p(\cdot),q(\cdot)}$ and $\|\cdot\|_{q(\cdot)}$ instead of $\|\cdot\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ and $\|\cdot\|_{L^{q(\cdot)}(0,\infty)}$, respectively.

First of all we establish very useful boundedness results for the anisotropic maximal function M_{HL} on variable exponent Lorentz spaces.

Proposition 2.1. *Assume that $p, q \in \mathbb{P}_1(0, \infty)$. Then, the maximal function M_{HL} is bounded from $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself.*

Proof. This property can be proved as [22, Theorem 3.12]. Indeed, it is clear that $\|M_{HL}f\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)}$, $f \in L^\infty(\mathbb{R}^n)$. On the other hand, according to [4, p. 14] M_{HL} is bounded from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. Then, by proceeding as in the proof of [3, Theorem 3.8, p. 122] we deduce that, for some $C > 0$, $(M_{HL}f)^* \leq Cf^{**}$. The proof now can be finished by using [22, Theorem 2.2]. \square

The following vectorial boundedness result for M_{HL} appears as Proposition 1.3 in the introduction.

Proposition 2.2. *Assume that $p, q \in \mathbb{P}_1$. For every $r \in (1, \infty)$ there exists $C > 0$ such that*

$$(1) \quad \left\| \left(\sum_{j \in \mathbb{N}} (M_{HL}(f_j))^r \right)^{1/r} \right\|_{p(\cdot),q(\cdot)} \leq C \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^r \right)^{1/r} \right\|_{p(\cdot),q(\cdot)},$$

for each sequence $(f_j)_{j \in \mathbb{N}}$ of functions in $L^1_{loc}(\mathbb{R}^n)$.

Proof. According to [5, Proposition 2.6, (ii)] the family of anisotropic balls $\{x + B_k\}_{x \in \mathbb{R}^n, k \in \mathbb{Z}}$ constitutes a Muckenhoupt basis in \mathbb{R}^n . For every $r > 0$, we define the r -power of the space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n))^r$, as follows

$$(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n))^r = \{f \text{ measurable in } \mathbb{R}^n : |f|^r \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)\},$$

(see [16, p. 67]). By using [14, Lemma 2.3] we deduce that, for every $r > 0$, $(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n))^r = \mathcal{L}^{rp(\cdot),rq(\cdot)}(\mathbb{R}^n)$.

We choose $\beta \in (0, 1)$ such that $\beta p, \beta q \in \mathbb{P}_1$. According to [22, Lemma 2.7], $(\mathcal{L}^{\beta p(\cdot),\beta q(\cdot)}(\mathbb{R}^n))^* = (\mathcal{L}^{\beta p(\cdot),\beta q(\cdot)}(\mathbb{R}^n))' = \mathcal{L}^{(\beta p(\cdot))',(\beta q(\cdot))'}(\mathbb{R}^n)$, where the first space represents the associate dual space of $\mathcal{L}^{\beta p(\cdot),\beta q(\cdot)}(\mathbb{R}^n)$ in the Kothe sense. Since $\beta p, \beta q \in \mathbb{P}_1$, Proposition 2.1 implies that M_{HL} is bounded from $\mathcal{L}^{(\beta p(\cdot))',(\beta q(\cdot))'}(\mathbb{R}^n)$ into itself. According to [16, Corollary 4.8 and Remark 4.9] we conclude that (1) holds for every $r \in (1, \infty)$. \square

As in [4, p. 44] we consider the following maximal functions that will be useful in the sequel. If $K \in \mathbb{Z}$ and $N, L \in \mathbb{N}$ we define, for every $f \in S'(\mathbb{R}^n)$,

$$M_\varphi^{0,K,L}(f)(x) = \sup_{k \in \mathbb{Z}, k \leq K} |(f * \varphi_k)(x)| \max(1, \rho(A^{-K}x))^{-L} (1 + b^{-k-K})^{-L}, \quad x \in \mathbb{R}^n,$$

$$M_\varphi^{K,L}(f)(x) = \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in x + B_k} |(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L} (1 + b^{-k-K})^{-L}, \quad x \in \mathbb{R}^n,$$

$$T_\varphi^{N,K,L}(f)(x) = \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in \mathbb{R}^n} \frac{|(f * \varphi_k)(y)|}{\max(1, \rho(A^{-k}(x-y)))^N} \frac{(1 + b^{-k-K})^{-L}}{\max(1, \rho(A^{-K}y))^L}, \quad x \in \mathbb{R}^n,$$

$$M_N^{0,K,L}(f) = \sup_{\varphi \in S_N} M_\varphi^{0,K,L}(f),$$

and

$$M_N^{K,L}(f) = \sup_{\varphi \in S_N} M_\varphi^{K,L}(f).$$

Now we are going to establish some properties we will need later.

Lemma 2.1. *Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$, $r > 0$ and $\varphi \in S(\mathbb{R}^n)$. Then, there exists a constant $C > 0$ which does not depend neither on K, L, N, r nor φ such that, for every $f \in S'(\mathbb{R}^n)$*

$$(T_\varphi^{N,K,L}(f)(x))^r \leq CM_{HL} \left((M_\varphi^{K,L}(f))^r \right)(x), \quad x \in \mathbb{R}^n.$$

Proof. Our proof is inspired on the ideas presented in [38, p. 10].

Let $f \in S'(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $k \leq K$ and $x \in \mathbb{R}^n$. Since

$$(|(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L} (1 + b^{-k-K})^{-L})^r \leq (M_\varphi^{K,L}(f))^r(z), \quad y \in z + B_k,$$

we can write

$$(|(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L} (1 + b^{-k-K})^{-L})^r \leq \frac{1}{|y + B_k|} \int_{y+B_k} (M_\varphi^{K,L}(f)(z))^r dz, \quad y \in \mathbb{R}^n.$$

Suppose that $z \in y + B_k$ and $y \in \mathbb{R}^n$. According to [4, p. 8] we have that $\rho(z - x) \leq b^\omega(\rho(z - y) + \rho(y - x)) \leq b^{\omega+k}(1 + b^{-k}\rho(y - x))$, where ω is the smallest integer so that $2B_0 \subset B_\omega$. We choose $s \in \mathbb{Z}$ such that $b^{\omega+k}(1 + b^{-k}\rho(y - x)) \in [b^s, b^{s+1})$. Then, we get

$$\begin{aligned} & (|(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L} (1 + b^{-k-K})^{-L})^r \\ & \leq b^\omega(1 + b^{-k}\rho(y - x)) \frac{1}{b^{\omega+k}(1 + b^{-k}\rho(y - x))} \int_{y+B_k} (M_\varphi^{K,L}(f)(z))^r dz \\ & \leq b^\omega(1 + b^{-k}\rho(y - x)) \frac{1}{b^s} \int_{x+B_{s+1}} (M_\varphi^{K,L}(f)(z))^r dz \\ & \leq 2b^{\omega+1}(1 + b^{-k}\rho(y - x))^{Nr} M_{HL} \left((M_\varphi^{K,L}(f))^r \right)(x), \quad y \in x + B_k. \end{aligned}$$

Hence, we obtain

$$(T_\varphi^{N,K,L}(f)(x))^r \leq CM_{HL} \left((M_\varphi^{K,L}(f))^r \right)(x), \quad x \in \mathbb{R}^n.$$

□

According to [4, p. 14], for every $1 < p < \infty$, the Hardy-Littlewood maximal function M_{HL} is bounded from $L^p(\mathbb{R}^n)$ into itself. So from Lemma 2.1 we deduce that, for every $1 < p < \infty$, there exists $C > 0$ such that

$$\|T_\varphi^{N,K,L}(f)\|_{L^p(\mathbb{R}^n)} \leq C \|M_\varphi^{K,L}(f)\|_{L^p(\mathbb{R}^n)}, \quad f \in S'(\mathbb{R}^n).$$

This property was proved in [4, Lemma 7.4] by using other procedure.

Lemma 2.2. *Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$ and $\varphi \in S(\mathbb{R}^n)$. Assume that $p, q \in \mathbb{P}_0$. Then,*

$$\|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} \leq C \|M_\varphi^{K,L}(f)\|_{p(\cdot),q(\cdot)}, \quad f \in S'(\mathbb{R}^n),$$

where $C > 0$ does not depend on (N, K, L, φ) .

Proof. We choose $r > 0$ such that $rp, rq \in \mathbb{P}_1$. Let $f \in S'(\mathbb{R}^n)$. According to [14, Lemma 2.3] and a well known property of the nondecreasing equimeasurable rearrangement we get

$$\begin{aligned} \|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} &= \|t^{\frac{1}{rp(t)} - \frac{1}{rq(t)}} ([T_\varphi^{N,K,L}(f)]^*(t))^{1/r}\|_{rq(\cdot)}^r \\ &= \|t^{\frac{1}{rp(t)} - \frac{1}{rq(t)}} \left[(T_\varphi^{N,K,L}(f))^{1/r} \right]^*(t)\|_{rq(\cdot)}^r \\ &= \| (T_\varphi^{N,K,L}(f))^{1/r} \|_{rp(\cdot),rq(\cdot)}^r. \end{aligned}$$

From Lemma 2.1 and Proposition 2.1 it follows that

$$\|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} \leq C \| (M_\varphi^{K,L}(f))^{1/r} \|_{rp(\cdot),rq(\cdot)}^r = C \|M_\varphi^{K,L}(f)\|_{p(\cdot),q(\cdot)}$$

□

Next two results were established in [4, p. 45-47] as Lemmas 7.5 and 7.6, respectively.

Lemma 2.3. *For every $N, L \in \mathbb{N}$, there exists $M_0 \in \mathbb{N}$ satisfying the following property: if $\varphi \in S(\mathbb{R}^n)$ is such that $\int \varphi(x)dx \neq 0$, then there exists $C > 0$ such that, for every $f \in S'(\mathbb{R}^n)$ and $K \in \mathbb{N}$,*

$$M_{M_0}^{0,K,L}(f)(x) \leq CT_\varphi^{N,K,L}(f)(x), \quad x \in \mathbb{R}^n.$$

Lemma 2.4. *Let $\varphi \in S(\mathbb{R}^n)$. Then, for every $M, K \in \mathbb{N}$ and $f \in S'(\mathbb{R}^n)$ there exist $L \in \mathbb{N}$ and $C > 0$ such that*

$$M_\varphi^{K,L}(f)(x) \leq C \max(1, \rho_A(x))^{-M}, \quad x \in \mathbb{R}^n.$$

Actually, L does not depend on $K \in \mathbb{N}$.

Lemma 2.5. *Let $p, q \in \mathbb{P}_0$. There exists $\alpha_0 > 0$ such that the function g_α defined by*

$$g_\alpha(x) = (\max(1, \rho_A(x)))^{-\alpha}, \quad x \in \mathbb{R}^n,$$

is in $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$, for every $\alpha \geq \alpha_0$.

Proof. Let $\alpha > 0$. According to [4, Lemma 3.2] we have that

$$g_\alpha(x) \leq h_\alpha(x) = C \begin{cases} 1, & |x| \leq 1, \\ |x|^{-\alpha \ln b / \ln \lambda_+}, & |x| > 1, \end{cases}$$

for certain $C > 0$. Here λ_+ is greater than $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ (for instance we can take $\lambda_+ = 2 \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$). Note that $g_\alpha^* \leq h_\alpha^*$.

To simplify we denote $v_n = |B(0, 1)|$. We have that

$$\mu_{h_\alpha}(s) = \begin{cases} 0, & s \geq C, \\ v_n(C/s)^{n \ln(\lambda_+)/(\alpha \ln(b))}, & s \in (0, C). \end{cases}$$

Then,

$$h_\alpha^*(t) = C \begin{cases} 1, & t \in (0, v_n), \\ (v_n/t)^{\alpha \ln(b)/(n \ln(\lambda_+))}, & t \geq v_n. \end{cases}$$

Since $q(0) > 0$ and $p(0) > 0$ we have that $\int_0^{v_n} t^{q(t)/p(t)-1} |g_\alpha^*(t)|^{q(t)} dt < \infty$. Also, there exists $\alpha_0 > 0$ such that $\int_{v_n}^\infty t^{q(t)/p(t)-1} |g_{\alpha_0}^*(t)|^{q(t)} dt < \infty$ because $p, q \in \mathbb{P}_0$.

Hence, $g_\alpha \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$, for every $\alpha \geq \alpha_0$. □

Lemma 2.6. *Let $p, q \in \mathfrak{P}_0$ and let D be a subset of \mathbb{R}^n . Then, $\chi_D \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ if, and only if, $|D| < \infty$.*

Proof. We have that $(\chi_D)^* = \chi_{[0, |D|]}$. Since $p, q \in \mathfrak{P}_0$, for every $\lambda > 0$

$$\int_0^\infty \left(\frac{(\chi_D)^*(t) t^{\frac{1}{p(t)} - \frac{1}{q(t)}}}{\lambda} \right)^{q(t)} dt = \int_0^{|D|} \frac{t^{-1+q(t)/p(t)}}{\lambda^{q(t)}} dt < \infty,$$

if, and only if, $|D| < \infty$. □

Proof of Theorem 1.1.

We recall that we are taking $f \in S'(\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^n)$ such that $\int \varphi(x) dx \neq 0$. It is clear that, for every $N \in \mathbb{N}$,

$$\|M_\varphi^0(f)\|_{p(\cdot), q(\cdot)} \leq \|M_\varphi(f)\|_{p(\cdot), q(\cdot)} \leq C \|f\|_{H_N^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}.$$

Hence, (i) \Rightarrow (ii) \Rightarrow (iii).

Now, we are going to complete the proof. Let M_0 be the value in Lemma 2.3 for $N = L = 0$. Then, for a certain $C > 0$,

$$(2) \quad \|M_M^0(g)\|_{p(\cdot), q(\cdot)} \leq C \|M_\varphi(g)\|_{p(\cdot), q(\cdot)}, \quad g \in S'(\mathbb{R}^n) \text{ and } M \geq M_0.$$

Indeed, by Lemma 2.3, there exists $C > 0$ such that

$$M_M^{0, K, 0}(g)(x) \leq C T_\varphi^{0, K, 0}(g)(x), \quad x \in \mathbb{R}^n, \quad g \in S'(\mathbb{R}^n), \quad K \in \mathbb{N}, \text{ and } M \geq M_0.$$

Then, Lemma 2.2 leads to

$$\|M_M^{0, K, 0}(g)\|_{p(\cdot), q(\cdot)} \leq C \|M_\varphi^{K, 0}(g)\|_{p(\cdot), q(\cdot)}, \quad g \in S'(\mathbb{R}^n), \quad K \in \mathbb{N} \text{ and } M \geq M_0.$$

By using monotone convergence theorem in $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ (see [22, Definition 2.5 v]) jointly with [14, Lemma 2.3] and by letting $K \rightarrow \infty$ we conclude that (2) holds.

Our next objective is to see that, for a certain $C > 0$,

$$(3) \quad \|M_\varphi(f)\|_{p(\cdot), q(\cdot)} \leq C \|M_\varphi^0(f)\|_{p(\cdot), q(\cdot)}.$$

Note that by combining (2), (3) and [4, Proposition 3.10] we conclude that (iii) \Rightarrow (ii) \Rightarrow (i).

In order to show (3) we firstly note that there exists $L_0 \in \mathbb{N}$ such that $M_\varphi^{K,L_0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for every $K \in \mathbb{N}$. Indeed, we denote by α_0 the constant appearing in Lemma 2.5. According to Lemma 2.4 we can find $L_0 \in \mathbb{N}$ such that, for every $K \in \mathbb{N}$, there exists $C > 0$ for which

$$(4) \quad M_\varphi^{K,L_0}(f)(x) \leq C \max(1, \rho(x))^{-\alpha_0}, \quad x \in \mathbb{R}^n.$$

Then, Lemma 2.5 leads to $M_\varphi^{K,L_0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for each $K \in \mathbb{N}$.

From Lemmas 2.2 and 2.3 we infer that there exist $M_0 \in \mathbb{N}$ and $C_0 > 0$ such that

$$(5) \quad \|M_{M_0}^{0,K,L_0}(f)\|_{p(\cdot),q(\cdot)} \leq C_0 \|M_\varphi^{K,L_0}(f)\|_{p(\cdot),q(\cdot)},$$

for every $K \in \mathbb{N}$.

Fix $K_0 \in \mathbb{N}$. We define the set Ω_0 by

$$\Omega_0 = \left\{ x \in \mathbb{R}^n : M_{M_0}^{0,K_0,L_0}(f)(x) \leq C_2 M_\varphi^{K_0,L_0}(f)(x) \right\},$$

where $C_2 > 0$ will be specified later.

By using (5), [14, Lemma 2.3] and [22, Theorem 2.4] and choosing $r > 1$ such that $rp, rq \in \mathbb{P}_1$, we get

$$\begin{aligned} \|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} &= \|t^{\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}} (M_\varphi^{K_0,L_0}(f))^*(t)\|_{q(\cdot)} \\ &= \|t^{\frac{1}{rp(\cdot)} - \frac{1}{rq(\cdot)}} ([M_\varphi^{K_0,L_0}(f)]^*(t))^{1/r}\|_{rq(\cdot)}^r \\ &= \|t^{\frac{1}{rp(\cdot)} - \frac{1}{rq(\cdot)}} ([M_\varphi^{K_0,L_0}(f)]^{1/r})^*(t)\|_{rq(\cdot)}^r = \|[M_\varphi^{K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}^r \\ &\leq \left(\|[M_\varphi^{K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)} \right)^r \\ &\leq A_1 \left\{ \left(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)} \right)^r + \left(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0^c}]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)} \right)^r \right\} \\ &\leq A_1 \left\{ \left(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)} \right)^r + \frac{1}{C_2} \left(\|[M_{M_0}^{0,K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)} \right)^r \right\} \\ &\leq A_2 \left\{ \left(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}]^{1/r}\|_{rp(\cdot),rq(\cdot)} \right)^r + \frac{1}{C_2} \left(\|[M_{M_0}^{0,K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)} \right)^r \right\} \\ &\leq A_2 \left(\|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)} + \frac{1}{C_2} \|M_{M_0}^{0,K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} \right) \\ &\leq A_2 \left(\|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)} + \frac{C_0}{C_2} \|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} \right), \end{aligned}$$

where $A_1, A_2 > 0$ depend only on p, q and r . Hence, by taking $C_2 \geq 2C_0A_2$ we obtain

$$\|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} \leq 2A_2 \|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)},$$

because $M_\varphi^{K_0,L_0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

According to [4, (7.16)] we have that

$$(6) \quad M_\varphi^{K_0,L_0}(f)(x) \leq C \left[M_{HL} \left(M_\varphi^{0,K_0,L_0}(f)^{1/r} \right) (x) \right]^r, \quad x \in \Omega_0.$$

The constant $C > 0$ does not depend on K_0 but it depends on L_0 .

From (6), Proposition 2.1 and [14, Lemma 2.3] we obtain

$$\begin{aligned} \|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)} &\leq C \left\| \left(M_{HL} (M_\varphi^{0,K_0,L_0}(f)^{1/r}) \right)^r \right\|_{p(\cdot),q(\cdot)} \\ &= C \|M_{HL}(M_\varphi^{0,K_0,L_0}(f)^{1/r})\|_{rp(\cdot),rq(\cdot)}^r \\ &\leq C \|M_\varphi^{0,K_0,L_0}(f)^{1/r}\|_{rp(\cdot),rq(\cdot)}^r \\ &= C \|M_\varphi^{0,K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}. \end{aligned}$$

We conclude that

$$\|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} \leq C \|M_\varphi^{0,K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}.$$

Note that again this constant $C > 0$ does not depend on K_0 and it depends on L_0 .

We have that $M_\varphi^{K,L_0}(f)(x) \uparrow M_\varphi(f)(x)$, as $K \rightarrow \infty$, for every $x \in \mathbb{R}^n$, and $M_\varphi^{0,K,L_0}(f)(x) \uparrow M_\varphi^0(f)(x)$, as $K \rightarrow \infty$, for every $x \in \mathbb{R}^n$. Hence, monotone convergence theorem in $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ -setting ([22, Theorem 2.8 and Definition 2.5, v]), jointly with [14, Lemma 2.3]), leads to

$$\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq C \|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)}.$$

Observe that the last inequality says that $M_\varphi(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, but the constant $C > 0$ depends on f because L_0 depends also on f .

On the other hand, since $M_\varphi(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $M_\varphi^{K,0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for every $K \in \mathbb{N}$. Hence, we can take $L_0 = 0$ at the beginning of the proof of this part. By proceeding as above we concluded that

$$\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq C \|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)},$$

where $C > 0$ does not depend on f .

Thus the proof of the theorem is finished. \square

The last part of this section is dedicated to establish some properties of the space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

Proposition 2.3. *Let $p, q \in \mathbb{P}_0$. Then, $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is continuously contained in $S'(\mathbb{R}^n)$.*

Proof. Let $f \in S'(\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^n)$. We define $\lambda_0 = |\langle f, \varphi \rangle|$. We can write

$$\lambda_0 = |(f * \varphi)(0)| \leq \sup_{z \in x+B_0} |(f * \varphi)(z)| \leq M_\varphi(f)(x), \quad x \in B_0.$$

Then,

$$|\{x \in \mathbb{R}^n : M_\varphi(f)(x) > \lambda_0/2\}| \geq 1,$$

and

$$(M_\varphi(f))^*(t) \geq \lambda_0/2, \quad t \in (0, 1).$$

Hence, we get

$$\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \geq \|t^{\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}} (M_\varphi(f))^*(t) \chi_{(1/2,1)}(t)\|_{q(\cdot)} \geq \frac{\lambda_0}{2} \|t^{\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}} \chi_{(1/2,1)}(t)\|_{q(\cdot)}.$$

Since $\|t^{\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}} \chi_{(1/2,1)}(t)\|_{q(\cdot)} > 0$ we conclude the desired result. \square

Proposition 2.4. *Let $p, q \in \mathbb{P}_0$. If $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, then f is a bounded distribution in $S'(\mathbb{R}^n)$.*

Proof. Let $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $\varphi \in S(\mathbb{R}^n)$. For every $x \in \mathbb{R}^n$, we have that

$$|(f * \varphi)(x)| \leq \sup_{z \in y+B_0} |(f * \varphi)(z)| \leq M_\varphi(f)(y), \quad y \in x + B_0.$$

By proceeding as in the proof of Proposition 2.3, we deduce that, for a certain $C > 0$,

$$|(f * \varphi)(x)| \leq C \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)}, \quad x \in \mathbb{R}^n.$$

Thus, we prove that f is a bounded distribution in $S'(\mathbb{R}^n)$. \square

Proposition 2.5. *Assume that $p, q \in \mathbb{P}_0$. Then, $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete.*

Proof. We choose $r \in (0, 1]$ such that $\frac{p(\cdot)}{r}, \frac{q(\cdot)}{r} \in \mathbb{P}_1$. In order to see that $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete it is sufficient to prove that if $(f_k)_{k \in \mathbb{N}}$ is a sequence in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ such that $\sum_{k \in \mathbb{N}} \|f_k\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)}^r < \infty$, then the series $\sum_{k \in \mathbb{N}} f_k$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ (see, for instance [3, Theorem 1.6, p.5]). As-

sume that $(f_k)_{k \in \mathbb{N}}$ is a sequence in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ such that $\sum_{k \in \mathbb{N}} \|f_k\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)}^r < \infty$. We

define, for every $j \in \mathbb{N}$, $F_j = \sum_{k=0}^j f_k$. According to [14, Lemma 2.3] and [22, Theorem 2.4], if

$j, \ell \in \mathbb{N}, j < \ell$, we get

$$\begin{aligned}
\|F_\ell - F_j\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}^r &= \left\| \sum_{k=j+1}^{\ell} f_k \right\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}^r \\
&\leq \left\| \sum_{k=j+1}^{\ell} M_N(f_k) \right\|_{p(\cdot), q(\cdot)}^r \\
&= \left\| \left(\sum_{k=j+1}^{\ell} M_N(f_k) \right)^r \right\|_{p(\cdot)/r, q(\cdot)/r} \\
&\leq \left\| \sum_{k=j+1}^{\ell} (M_N(f_k))^r \right\|_{p(\cdot)/r, q(\cdot)/r} \\
&\leq \left\| \sum_{k=j+1}^{\ell} (M_N(f_k))^r \right\|_{p(\cdot)/r, q(\cdot)/r}^{(1)} \\
&\leq \sum_{k=j+1}^{\ell} \left\| (M_N(f_k))^r \right\|_{p(\cdot)/r, q(\cdot)/r}^{(1)} \\
&\leq C \sum_{k=j+1}^{\ell} \left\| (M_N(f_k))^r \right\|_{p(\cdot)/r, q(\cdot)/r} \\
&= C \sum_{k=j+1}^{\ell} \|M_N(f_k)\|_{p(\cdot), q(\cdot)}^r \\
&= C \sum_{k=j+1}^{\ell} \|f_k\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}^r.
\end{aligned}$$

Hence, $(F_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. By Proposition 2.3 $(F_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $S'(\mathbb{R}^n)$. Then, there exists $F \in S'(\mathbb{R}^n)$ such that $F_j \rightarrow F$, as $j \rightarrow \infty$, in $S'(\mathbb{R}^n)$. We have that

$$M_N(F) \leq \lim_{j \rightarrow \infty} \sum_{k=0}^j M_N(f_k).$$

According to [22, Theorem 2.8 and Definition 2.5 v)] by proceeding as above we obtain

$$\begin{aligned}
\|M_N(F)\|_{p(\cdot), q(\cdot)}^r &\leq \left\| \lim_{j \rightarrow \infty} \sum_{k=0}^j M_N(f_k) \right\|_{p(\cdot), q(\cdot)}^r \\
&= \lim_{j \rightarrow \infty} \left\| \sum_{k=0}^j M_N(f_k) \right\|_{p(\cdot), q(\cdot)}^r \\
&\leq \sum_{k \in \mathbb{N}} \left\| (M_N(f_k))^r \right\|_{p(\cdot)/r, q(\cdot)/r} \\
&= C \sum_{k \in \mathbb{N}} \|f_k\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}^r.
\end{aligned}$$

Then, $F \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. Also, we have that

$$\left\| F - \sum_{k=0}^j f_k \right\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}^r \leq C \sum_{k=j+1}^{\infty} \|f_k\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}^r, \quad j \in \mathbb{N}.$$

Hence, $F = \sum_{k \in \mathbb{N}} f_k$ in the sense of convergence in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. □

3. A CALDERÓN-ZYGMUND DECOMPOSITION

In this section we study a Calderón-Zygmund decomposition for our anisotropic setting (associated with the matrix dilation A) for a distribution $f \in S'(\mathbb{R}^n)$ satisfying that $|\{x \in \mathbb{R}^n : M_N f(x) > \lambda\}| < \infty$, where $N \in \mathbb{N}$, $N \geq 2$ and $\lambda > 0$. We will use the ideas and results established in [4, Section 5, Chapter I]. Also we prove new properties involving variable exponent Hardy-Lorentz norms that will be useful in the sequel.

Let $\lambda > 0$, $N \in \mathbb{N}$, $N \geq 2$ and $f \in S'(\mathbb{R}^n)$ such that $|\Omega_\lambda| < \infty$ where

$$\Omega_\lambda = \{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}.$$

According to the Whitney Lemma ([4, Lemma 2.7]) there exist sequences $(x_j)_{j \in \mathbb{N}} \subset \Omega_\lambda$ and $(\ell_j)_{j \in \mathbb{N}} \subset \mathbb{Z}$ satisfying that

$$(7) \quad \Omega_\lambda = \bigcup_{j \in \mathbb{N}} (x_j + B_{\ell_j});$$

$$(8) \quad (x_i + B_{\ell_i - \omega}) \cap (x_j + B_{\ell_j - \omega}) = \emptyset, \quad i, j \in \mathbb{N}, \quad i \neq j;$$

$$(9) \quad (x_j + B_{\ell_j + 4\omega}) \cap \Omega_\lambda^c = \emptyset, \quad (x_j + B_{\ell_j + 4\omega + 1}) \cap \Omega_\lambda^c \neq \emptyset, \quad j \in \mathbb{N};$$

$$(10) \quad \text{if } i, j \in \mathbb{N} \text{ and } (x_i + B_{\ell_i + 2\omega}) \cap (x_j + B_{\ell_j + 2\omega}) \neq \emptyset, \text{ then } |\ell_i - \ell_j| \leq \omega;$$

$$(11) \quad \sharp\{j \in \mathbb{N} : (x_i + B_{\ell_i + 2\omega}) \cap (x_j + B_{\ell_j + 2\omega}) \neq \emptyset\} \leq L, \quad i \in \mathbb{N}.$$

Here L denotes a nonnegative integer that does not depend on Ω_λ . If $E \subset \mathbb{R}^n$ by $\sharp E$ we represent the cardinal of E .

Assume now that $\theta \in C^\infty(\mathbb{R}^n)$ verifies that $\text{supp } \theta \subset B_\omega$, $0 \leq \theta \leq 1$, and $\theta = 1$ on B_0 . We define, for every $j \in \mathbb{N}$,

$$\theta_j(x) = \theta(A^{-\ell_j}(x - x_j)), \quad x \in \mathbb{R}^n,$$

and, for every $i \in \mathbb{N}$,

$$\zeta_i(x) = \begin{cases} \theta_i(x) / (\sum_{j \in \mathbb{N}} \theta_j(x)), & x \in \Omega_\lambda, \\ 0, & x \in \Omega_\lambda^c. \end{cases}$$

The sequence $\{\zeta_i\}_{i \in \mathbb{N}}$ is a smooth partition of unity associated with the covering $\{x_i + B_{\ell_i + \omega}\}_{i \in \mathbb{N}}$ of Ω .

Let $i, s \in \mathbb{N}$. By \mathcal{P}_s we denote the linear space of polynomials in \mathbb{R}^n with degree at most s . \mathcal{P}_s is endowed with the norm $\|\cdot\|_{i,s}$ defined by

$$\|P\|_{i,s} = \left(\frac{1}{\int \zeta_i} \int_{\mathbb{R}^n} |P(x)|^2 \zeta_i(x) dx \right)^{1/2}, \quad P \in \mathcal{P}_s.$$

Thus $(\mathcal{P}_s, \|\cdot\|_{i,s})$ is a Hilbert space. We consider on \mathcal{P}_s the functional $T_{f,i,s}$ given by

$$T_{f,i,s}(Q) = \frac{1}{\int \zeta_i} \langle f, Q \zeta_i \rangle, \quad Q \in \mathcal{P}_s.$$

$T_{f,i,s}$ is continuous in $(\mathcal{P}_s, \|\cdot\|_{i,s})$ and there exists $P_{f,i,s} \in \mathcal{P}_s$ such that

$$T_{f,i,s}(Q) = \frac{1}{\int \zeta_i} \int_{\mathbb{R}^n} P_{f,i,s}(x) Q(x) \zeta_i(x) dx, \quad Q \in \mathcal{P}_s.$$

To simplify we write P_i to refer to $P_{f,i,s}$. We define $b_i = (f - P_i) \zeta_i$.

We will find values of s and N for which the series $\sum_{i \in \mathbb{N}} b_i$ converges in $S'(\mathbb{R}^n)$ provided that $f \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. Then, we define $g = f - \sum_{i \in \mathbb{N}} b_i$.

The representation $f = g + \sum_{i \in \mathbb{N}} b_i$ is known as the Calderón-Zygmund decomposition of f of degree s and height λ associated with $M_N(f)$.

Note firstly that if $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $N \in \mathbb{N}$, $N \geq N_0$, then $\|\chi_{\{x \in \mathbb{R}^n: M_N(f)(x) > \mu\}}\|_{p(\cdot),q(\cdot)} < \infty$ for every $\mu > 0$, and by Lemma 2.6, $|\{x \in \mathbb{R}^n: M_N(f)(x) > \mu\}| < \infty$, for every $\mu > 0$. Here N_0 is the one defined in Theorem 1.1.

Our next objective is to prove that $L^1_{loc}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. This property will be useful to deal with the proof that every element of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ can be represented as a sum of a special kind of distributions, so called atoms, which will be developed in the next section.

We need to establish some auxiliary results. Firstly we prove the absolute continuity of the norm $\|\cdot\|_{p(\cdot),q(\cdot)}$.

Proposition 3.1. *Let $(E_k)_{k \in \mathbb{N}}$ be a sequence of measurable sets satisfying that $E_k \supset E_{k+1}$, $k \in \mathbb{N}$, $|E_1| < \infty$, and $|\cap_{k \in \mathbb{N}} E_k| = 0$. Assume that $p, q \in \mathbb{P}_0$. If $f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then*

$$\|f\chi_{E_k}\|_{p(\cdot),q(\cdot)} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Proof. Let $f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We have that $(f\chi_{E_k})^* \leq f^*$. Then, $f\chi_{E_k} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover, since $|\cap_{k \in \mathbb{N}} E_k| = \lim_{k \rightarrow \infty} |E_k| = 0$, for every $t > 0$ there exists $k_0 \in \mathbb{N}$ such that $(f\chi_{E_k})^*(t) = 0$, $k \in \mathbb{N}$, $k \geq k_0$. Hence, for every $t > 0$,

$$t^{\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}} (f\chi_{E_k})^*(t) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

By using dominated convergence theorem ([18, Lemma 3.2.8]) jointly with [14, Lemma 2.3] and by taking into account that $q \in \mathbb{P}_0$ and that $f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we obtain

$$\|f\chi_{E_k}\|_{p(\cdot),q(\cdot)} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

□

Note that the last property also holds by more general exponent functions p and q .

Proposition 3.2. *Assume that $p, q \in \mathbb{P}_0$. There exists $s_0 \in \mathbb{N}$, such that, for every $s \in \mathbb{N}$, $s \geq s_0$, and each $N \in \mathbb{N}$, $N > \max\{N_0, s\}$, where N_0 is defined in Theorem 1.1, the following two properties holds.*

(i) *Let $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $\lambda > 0$. If $f = g + \sum_{i \in \mathbb{N}} b_i$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N f$ of height λ and degree s , then the series $\sum_{i \in \mathbb{N}} b_i$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.*

(ii) *Suppose that $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and that, for every $j \in \mathbb{Z}$, $f = g_j + \sum_{i \in \mathbb{N}} b_{i,j}$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N f$ of height 2^j and degree s . Then, $(g_j)_{j \in \mathbb{Z}} \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $(g_j)_{j \in \mathbb{Z}}$ converges to f , as $j \rightarrow +\infty$, in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.*

Proof. (i) Let $s, N \in \mathbb{N}$, $N > \max\{N_0, s\}$. The Calderón-Zygmund decomposition of f associated to $M_N f$ of height $\lambda > 0$ and degree s is $f = g + \sum_{i \in \mathbb{N}} b_i$. We are going to specify s and N in order that the series $\sum_{i \in \mathbb{N}} b_i$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

By using [4, Lemmas 5.4 and 5.6] we get that there exists $C > 0$ so that, for every $i \in \mathbb{N}$,

$$M_N(b_i)(x) \leq C \left(M_N f(x) \chi_{x_i + B_{\ell_i + 2\omega}}(x) + \lambda \sum_{k \in \mathbb{N}} \lambda_-^{-k(s+1)} \chi_{x_i + (B_{\ell_i + 2\omega + 1 + k} \setminus B_{\ell_i + 2\omega + k})}(x) \right), \quad x \in \mathbb{R}^n.$$

Let $j, m \in \mathbb{N}$, $j < m$. We infer, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} M_N \left(\sum_{i=j}^m b_i \right) (x) &\leq \sum_{i=j}^m M_N(b_i)(x) \\ &\leq C \left(M_N f(x) \sum_{i=j}^m \chi_{x_i + B_{\ell_i + 2\omega}}(x) + \lambda \sum_{i=j}^m \sum_{k \in \mathbb{N}} \lambda_-^{-k(s+1)} \chi_{x_i + (B_{\ell_i + 2\omega + 1 + k} \setminus B_{\ell_i + 2\omega + k})}(x) \right). \end{aligned}$$

We also have that, for every $x \in x_i + (B_{\ell_i + 2\omega + 1 + k} \setminus B_{\ell_i + 2\omega + k})$, with $i, k \in \mathbb{N}$, $i \leq m$,

$$M_{HL} \left(\chi_{x_i + B_{\ell_i + 2\omega}} \right) (x) \geq \frac{1}{|x_i + B_{\ell_i + 2\omega + 1 + k}|} \int_{x_i + B_{\ell_i + 2\omega + 1 + k}} \chi_{x_i + B_{\ell_i + 2\omega}}(y) dy = b^{-k-1}.$$

We choose $r > 1$ such that $rp, rq \in \mathbb{P}_1$. Then, we take $s \in \mathbb{N}$ such that $\lambda_-^{-s} b^r \leq 1$ and $N_0 < s$. We get, for every $i \in \mathbb{N}$, $i \leq m$,

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda_-^{-k(s+1)} \chi_{x_i + (B_{\ell_i+2\omega+1+k} \setminus B_{\ell_i+2\omega+k})}(x) &\leq C \max_{k \in \mathbb{N}} (\lambda_-^{-s-1} b^r)^k \left(M_{HL} \left(\chi_{x_i + B_{\ell_i+2\omega}} \right) (x) \right)^r \\ &\leq C \left(M_{HL} \left(\chi_{x_i + B_{\ell_i+2\omega}} \right) (x) \right)^r, \quad x \in (x_i + B_{\ell_i+2\omega})^c. \end{aligned}$$

Hence we obtain

$$M_N \left(\sum_{i=j}^m b_i \right) (x) \leq C_0 \left(M_N f(x) \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}}(x) + \lambda \sum_{i=j}^m \left(M_{HL} \left(\chi_{x_i + B_{\ell_i+2\omega}} \right) (x) \right)^r \right), \quad x \in \mathbb{R}^n.$$

By using [14, Lemma 2.3], since $\mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ is a quasi Banach space we obtain

$$\begin{aligned} \left\| M_N \left(\sum_{i=j}^m b_i \right) \right\|_{p(\cdot), q(\cdot)} &\leq C \left(\left\| M_N(f) \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right\|_{p(\cdot), q(\cdot)} \right. \\ &\quad \left. + \lambda \left\| \sum_{i=j}^m \left(M_{HL} \left(\chi_{x_i + B_{\ell_i+2\omega}} \right) \right)^r \right\|_{p(\cdot), q(\cdot)} \right) \\ (12) \quad &= C \left(\left\| M_N(f) \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right\|_{p(\cdot), q(\cdot)} + \lambda \left\| \left(\sum_{i=j}^m \left(M_{HL} \left(\chi_{x_i + B_{\ell_i+2\omega}} \right) \right)^r \right)^{1/r} \right\|_{rp(\cdot), rq(\cdot)}^r \right) \end{aligned}$$

By using Proposition 2.2 we get

$$\begin{aligned} \left\| \left(\sum_{i=j}^m \left(M_{HL} \left(\chi_{x_i + B_{\ell_i+2\omega}} \right) \right)^r \right)^{1/r} \right\|_{rp(\cdot), rq(\cdot)}^r &\leq C \left\| \left(\sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right)^{1/r} \right\|_{rp(\cdot), rq(\cdot)}^r \\ &= C \left\| \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right\|_{p(\cdot), q(\cdot)}. \end{aligned}$$

From (11) and (12) it follows that

$$\begin{aligned} \left\| M_N \left(\sum_{i=j}^m b_i \right) \right\|_{p(\cdot), q(\cdot)} &\leq C \left(\left\| M_N(f) \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right\|_{p(\cdot), q(\cdot)} + \lambda \left\| \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right\|_{p(\cdot), q(\cdot)} \right) \\ &\leq C \left\| M_N(f) \sum_{i=j}^m \chi_{x_i + B_{\ell_i+2\omega}} \right\|_{p(\cdot), q(\cdot)} \\ &\leq C \left\| M_N(f) \chi_{\cup_{i=j}^{\infty} (x_i + B_{\ell_i+2\omega})} \right\|_{p(\cdot), q(\cdot)}. \end{aligned}$$

We define, for every $k \in \mathbb{N}$, $E_k = \cup_{i \in \mathbb{N}} (x_i + B_{\ell_i+2\omega})$. By (11) there exists $C > 0$ such that $\sum_{i=k}^{\infty} \chi_{x_i + B_{\ell_i+2\omega}} \leq C \chi_{E_k}$, $k \in \mathbb{N}$. By (7) and (8), $\cup_{i \in \mathbb{N}} (x_i + B_{\ell_i-\omega}) \subset \Omega_\lambda$, and then $\sum_{i \in \mathbb{N}} |x_i + B_{\ell_i-\omega}| = b^{-\omega} \sum_{i \in \mathbb{N}} b^{\ell_i} \leq |\Omega_\lambda| < \infty$, where $\Omega_\lambda = \{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}$. We deduce that

$$|E_k| \leq \sum_{i=k}^{\infty} |x_i + B_{\ell_i+2\omega}| = b^{2\omega} \sum_{i=k}^{\infty} b^{\ell_i}, \quad k \in \mathbb{N}.$$

Proposition 3.1 implies that

$$\lim_{k \rightarrow \infty} \|M_N(f) \chi_{E_k}\|_{p(\cdot), q(\cdot)} = 0.$$

Hence, the sequence $\{\sum_{i=0}^k b_i\}_{k \in \mathbb{N}}$ is Cauchy in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. Since $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ is complete (Proposition 2.5), the series $\sum_{i \in \mathbb{N}} b_i$ converges in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.

(ii) In order to prove this property we can proceed as in the proof of (i). Assume that $j \in \mathbb{Z}$. We define $\Omega_j = \{x \in \mathbb{R}^n : M_N f(x) > 2^j\}$. By putting $b_j = \sum_{i \in \mathbb{N}} b_{i,j}$, since, as we have just proved in

(i), the last series converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and then in $S'(\mathbb{R}^n)$, we obtain, for the chosen $r > 1$ verifying that $rp, rq \in \mathbb{P}_1$,

$$M_N(b_j)(x) \leq C_0 \left(M_N f(x) \chi_{\Omega_j}(x) + 2^j \sum_{i \in \mathbb{N}} \left(M_{HL} \left(\chi_{x_i + B_{\ell_i + 2\omega}} \right) (x) \right)^r \right), \quad x \in \mathbb{R}^n.$$

It follows that

$$(13) \quad \begin{aligned} & \|M_N(b_j)\|_{p(\cdot),q(\cdot)} \\ & \leq C \left(\|M_N(f) \chi_{\Omega_j}\|_{p(\cdot),q(\cdot)} + 2^j \left\| \left(\sum_{i \in \mathbb{N}} \left(M_{HL} \left(\chi_{x_i + B_{\ell_i + 2\omega}} \right) \right)^r \right)^{1/r} \right\|_{rp(\cdot),rq(\cdot)}^r \right) \end{aligned}$$

From Proposition 2.2 we get

$$\begin{aligned} \left\| \left(\sum_{i \in \mathbb{N}} \left(M_{HL} \left(\chi_{\{x_i + B_{\ell_i + 2\omega}\}} \right) \right)^r \right)^{1/r} \right\|_{rp(\cdot),rq(\cdot)} & \leq C \left\| \left(\sum_{i \in \mathbb{N}} \chi_{x_i + B_{\ell_i + 2\omega}} \right)^{1/r} \right\|_{rp(\cdot),rq(\cdot)} \\ & = C \left\| \sum_{i \in \mathbb{N}} \chi_{x_i + B_{\ell_i + 2\omega}} \right\|_{p(\cdot),q(\cdot)} \\ & \leq C \| \chi_{\Omega_j} \|_{p(\cdot),q(\cdot)}. \end{aligned}$$

From (13) it follows that

$$\begin{aligned} \|M_N(b_j)\|_{p(\cdot),q(\cdot)} & \leq C \left(\|M_N(f) \chi_{\Omega_j}\|_{p(\cdot),q(\cdot)} + 2^j \| \chi_{\Omega_j} \|_{p(\cdot),q(\cdot)} \right) \\ & \leq C \|M_N(f) \chi_{\Omega_j}\|_{p(\cdot),q(\cdot)}. \end{aligned}$$

Since $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, by invoking again [14, Lemma 2.3], we have that

$$\|M_N(f)\|_{p(\cdot),q(\cdot)}^{1/r} = \|(M_N(f))^{1/r}\|_{rp(\cdot),rq(\cdot)} < \infty.$$

Then, by [22, Theorem 2.8] (see [22, Definition 2.5 vii]), $M_N(f)(x) < \infty$, a.e. $x \in \mathbb{R}^n$. Hence, $M_N(f) \chi_{\Omega_j} \downarrow 0$, as $j \rightarrow +\infty$, for a.e. $x \in \mathbb{R}^n$. According to Proposition 3.1 we conclude that $\|M_N(f) \chi_{\Omega_j}\|_{p(\cdot),q(\cdot)} \rightarrow 0$, as $j \rightarrow +\infty$. Hence, $\|M_N(b_j)\|_{p(\cdot),q(\cdot)} \rightarrow 0$, as $j \rightarrow +\infty$, and $\|f - g_j\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)} \rightarrow 0$, as $j \rightarrow +\infty$.

The proof of this Proposition is now finished. \square

By $C_c^\infty(\mathbb{R}^n)$ we denote the space of smooth functions with compact support in \mathbb{R}^n . We say that a distribution $h \in S'(\mathbb{R}^n)$ is in $L_{loc}^1(\mathbb{R}^n)$ when there exists a (unique) $H \in L_{loc}^1(\mathbb{R}^n)$ such that

$$\langle h, \phi \rangle = \int_{\mathbb{R}^n} H(x) \phi(x) dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

The space $S'(\mathbb{R}^n) \cap L_{loc}^1(\mathbb{R}^n)$ is also sometimes denoted by $S_r(\mathbb{R}^n)$ and it was studied, for instance, in [19], [46] and [47].

Proposition 3.3. *If $f \in S'(\mathbb{R}^n)$, $\lambda > 0$, $s, N \in \mathbb{N}$, $N \geq 2$ and $s < N$, and $f = g + \sum_{i \in \mathbb{N}} b_i$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N(f)$ of height λ and degree s , then $g \in L_{loc}^1(\mathbb{R}^n)$.*

Proof. Let $\lambda > 0$, $N \in \mathbb{N}$, $N \geq 2$, $s \in \mathbb{N}$, $s < N$, and $f \in S'(\mathbb{R}^n)$ such that $|\Omega_\lambda| < \infty$ where $\Omega_\lambda = \{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}$. We write $f = g + \sum_{i \in \mathbb{N}} b_i$ the Calderón-Zygmund decomposition of f associated to $M_N(f)$ of height λ and degree s .

According to [4, Lemma 5.9] we have that

$$M_N(g)(x) \leq C \lambda \sum_{i \in \mathbb{N}} \lambda_-^{-t_i(s+1)} + M_N(f)(x) \chi_{\Omega_\lambda^c}(x), \quad x \in \mathbb{R}^n,$$

where

$$t_i = t_i(x) = \begin{cases} t, & \text{if } x \in x_i + (B_{\ell_i + 2\omega + t + 1} \setminus B_{\ell_i + 2\omega + t}), \text{ for some } t \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

As it was shown in the proof of [4, Lemma 5.10 (i), p. 34] we get

$$\int_{\mathbb{R}^n} \sum_{i \in \mathbb{N}} \lambda_-^{-t_i(x)(s+1)} dx \leq C|\Omega_\lambda|.$$

Then, since $M_N(f)(x) \leq \lambda$, $x \in \Omega_\lambda^c$, we obtain that $M_N(g) \in L_{loc}^1(\mathbb{R}^n)$.

Let $\varphi \in S(\mathbb{R}^n)$. Since for a certain $C > 0$ $g * \varphi_k \leq CM_N(g)$, $k \in \mathbb{N}$, by proceeding as in the proof of [4, Theorem 3.9] we can prove that, for every compact subset F of \mathbb{R}^n there exists a sequence $\{k_j\}_{j \in \mathbb{N}} \subset \mathbb{Z}$ such that $k_j \rightarrow -\infty$, as $j \rightarrow \infty$, and $g * \varphi_{k_j} \rightarrow G_F$, as $j \rightarrow \infty$, in the weak topology of $L^1(F)$ for a certain $G_F \in L^1(F)$. A diagonal argument allows us to get a sequence $\{k_j\}_{j \in \mathbb{N}} \subset \mathbb{Z}$ such that $k_j \rightarrow -\infty$, as $j \rightarrow \infty$, and $g * \varphi_{k_j} \rightarrow G$, in the weak $*$ topology of $\mathcal{M}(K)$ (the space of complex measures supported in K) for every compact subset K of \mathbb{R}^n , being $G \in L_{loc}^1(\mathbb{R}^n)$. According to [4, Lemma 3.8] $g * \varphi_{k_j} \rightarrow g$, as $j \rightarrow \infty$ in $S'(\mathbb{R}^n)$. If $\phi \in C_c^\infty(\mathbb{R}^n)$, we have that

$$(14) \quad \langle g, \phi \rangle = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} (g * \varphi_{k_j})(x) \phi(x) dx = \int_{\mathbb{R}^n} G(x) \phi(x) dx.$$

Since $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $S(\mathbb{R}^n)$, g is characterized by (14). \square

Corollary 3.1. *Assume that $p, q \in \mathbb{P}_0$. Then, $L_{loc}^1(\mathbb{R}^n) \cap H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ is a dense subspace of $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.*

Proof. This property is a consequence of Propositions 3.2 and 3.3. \square

We finish this section with a convergence property for the good parts of Calderón-Zygmund decomposition of distributions in $L_{loc}^1(\mathbb{R}^n) \cap H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ which we will use in the proof of atomic decompositions of the elements of $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.

Proposition 3.4. *Assume that $p, q \in \mathbb{P}_0$, and $f \in L_{loc}^1(\mathbb{R}^n) \cap H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. For every $j \in \mathbb{N}$, $f = g_j + \sum_{i \in \mathbb{N}} b_{i,j}$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N(f)$ of height 2^j and degree s , with $s, N \in \mathbb{N}$, $s \geq s_0$ and $N > \max\{s, N_0\}$, where N_0 is as in Theorem 1.1 and s_0 is as in Proposition 3.2. Then, $g_j \rightarrow 0$, as $j \rightarrow -\infty$, in $S'(\mathbb{R}^n)$.*

Proof. Since $f \in L_{loc}^1(\mathbb{R}^n)$, there exists a unique $F \in L_{loc}^1(\mathbb{R}^n)$ such that

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} F(x) \phi(x) dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

According to Proposition 3.3, for every $j \in \mathbb{Z}$, there exists a unique $G_j \in L_{loc}^1(\mathbb{R}^n)$ for which

$$(15) \quad \langle g_j, \phi \rangle = \int_{\mathbb{R}^n} G_j(x) \phi(x) dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

Let $j \in \mathbb{Z}$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. We are going to see that

$$\sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} |(F(x) - P_{i,j}(x)) \zeta_{i,j}(x)| |\phi(x)| dx < \infty.$$

For every $i \in \mathbb{N}$, by [4, Lemma 5.3] we have that

$$\begin{aligned} & \int_{\mathbb{R}^n} |(F(x) - P_{i,j}(x)) \zeta_{i,j}(x)| |\phi(x)| dx \\ & \leq C \left(\int_{(x_{i,j} + B_{l_{i,j} + \omega}) \cap \text{supp}(\phi)} |F(x)| dx + 2^j |(x_{i,j} + B_{l_{i,j} + \omega}) \cap \text{supp}(\phi)| \right). \end{aligned}$$

Then,

$$\sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} |(F(x) - P_{i,j}(x)) \zeta_{i,j}(x)| |\phi(x)| dx \leq C \left(\int_{\text{supp}(\phi)} |F(x)| dx + 2^j |\text{supp}(\phi)| \right).$$

Hence, from Proposition 3.2, (i), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{i \in \mathbb{N}} (F(x) - P_{i,j}(x)) \zeta_{i,j}(x) \phi(x) dx &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} (F(x) - P_{i,j}(x)) \zeta_{i,j}(x) \phi(x) dx \\ &= \sum_{i \in \mathbb{N}} \langle (f - P_{i,j}) \zeta_{i,j}, \phi \rangle \\ &= \langle \sum_{i \in \mathbb{N}} (f - P_{i,j}) \zeta_{i,j}, \phi \rangle. \end{aligned}$$

Then, there exists a measurable subset $E \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus E| = 0$, and

$$G_j(x) = F(x) - \sum_{i \in \mathbb{N}} (F(x) - P_{i,j}(x)) \zeta_{i,j}(x), \quad x \in E \text{ and } j \in \mathbb{Z},$$

for a suitable sense of the convergence of series. Note that we have used a diagonal argument to justify the convergence for every $j \in \mathbb{Z}$.

We can write

$$G_j(x) = F(x) \chi_{\Omega_j^c}(x) - \sum_{i \in \mathbb{N}} P_{i,j}(x) \zeta_{i,j}(x), \quad x \in E \text{ and } j \in \mathbb{Z},$$

where $\Omega_j = \{x \in \mathbb{R}^n : M_N(f)(x) > 2^j\}$, $j \in \mathbb{Z}$. Note that the last series is actually a finite sum for every $x \in \mathbb{R}^n$.

Let $j \in \mathbb{Z}$. According to [4, Lemma 5.3] we obtain

$$|G_j(x)| \leq C2^j, \quad \text{a.e. } x \in \Omega_j.$$

On the other hand, $G_j(x) = F(x)$, a.e. $x \in \Omega_j^c$. Also, we have that

$$|F| \leq \sup_{k \in \mathbb{Z}, \varphi \in C_c^\infty(\mathbb{R}^n) \cap S_N} |f * \varphi_k| \leq M_N(f).$$

Then, $|G_j(x)| \leq C2^j$, a.e. $x \in \Omega_j^c$. Hence, we conclude that

$$(16) \quad |G_j(x)| \leq C2^j, \quad \text{a.e. } x \in \mathbb{R}^n.$$

We consider the functional T_j defined on $S(\mathbb{R}^n)$ by

$$T_j(\phi) = \int_{\mathbb{R}^n} G_j(x) \phi(x) dx, \quad \phi \in S(\mathbb{R}^n).$$

From (16) we deduce that $T_j \in S'(\mathbb{R}^n)$. By (15), $T_j(\phi) = \langle g_j, \phi \rangle$, $\phi \in C_c^\infty(\mathbb{R}^n)$. Then,

$$\langle g_j, \phi \rangle = \int_{\mathbb{R}^n} G_j(x) \phi(x) dx, \quad \phi \in S(\mathbb{R}^n),$$

and, again from (16) it follows that $g_j \rightarrow 0$, as $j \rightarrow -\infty$, in $S'(\mathbb{R}^n)$. \square

4. ATOMIC CHARACTERIZATION (PROOF OF THEOREM 1.2)

As we mentioned in the introduction, we are going to prove Theorem 1.2 in two steps, firstly in the case that $r = \infty$ and then when $r < \infty$.

4.1. Proof of Theorem 1.2 when $r = \infty$. (i) Suppose that, for every $j \in \mathbb{N}$, a_j is a $(p(\cdot), q(\cdot), \infty, s)$ -atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$. Here $s \in \mathbb{N}$ will be fixed later. Assume also that $(\lambda_j)_{j \in \mathbb{N}} \subset (0, \infty)$ and that

$$(17) \quad \left\| \sum_{j \in \mathbb{N}} \frac{\lambda_j}{\|\chi_{x_j + B_{\ell_j}}\|_{p(\cdot), q(\cdot)}} \chi_{x_j + B_{\ell_j}} \right\|_{p(\cdot), q(\cdot)} < \infty.$$

We are going to show that the series $\sum_{j \in \mathbb{N}} \lambda_j a_j$ converges in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. Let $\ell, m \in \mathbb{N}$, $\ell < m$. We define

$$f_{\ell, m} = \sum_{j=\ell}^m \lambda_j a_j,$$

and we take $\varphi \in S(\mathbb{R}^n)$. We have that

$$\begin{aligned}
\|M_\varphi(f_{\ell,m})\|_{p(\cdot),q(\cdot)} &\leq \left\| \sum_{j=\ell}^m \lambda_j M_\varphi(a_j) \right\|_{p(\cdot),q(\cdot)} \\
&\leq C \left(\left\| \sum_{j=\ell}^m \lambda_j M_\varphi(a_j) \chi_{x_j+B_{\ell_j}+\omega} \right\|_{p(\cdot),q(\cdot)} + \left\| \sum_{j=\ell}^m \lambda_j M_\varphi(a_j) \chi_{x_j+B_{\ell_j}^c+\omega} \right\|_{p(\cdot),q(\cdot)} \right) \\
(18) \quad &= I_1 + I_2.
\end{aligned}$$

We now estimate I_i , $i = 1, 2$. Firstly we study I_1 . Let $j \in \mathbb{N}$. Since a_j is a $(p(\cdot), q(\cdot), \infty, s)$ -atom, we can write

$$M_\varphi(a_j)(x) \leq \|a_j\|_\infty \|\varphi\|_1 \leq C \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1}, \quad x \in \mathbb{R}^n.$$

By defining $g_j = \chi_{x_j+B_{\ell_j}} (\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \lambda_j)^\alpha$ it follows that

$$\begin{aligned}
M_{HL}g_j(x) &\geq (\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \lambda_j)^\alpha \frac{1}{|B_{\ell_j}+\omega|} \int_{x_j+B_{\ell_j}+\omega} \chi_{x_j+B_{\ell_j}}(y) dy \\
&= b^{-\omega} (\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \lambda_j)^\alpha, \quad x \in x_j + B_{\ell_j}+\omega,
\end{aligned}$$

where $\alpha \in (0, 1)$ is such that $p(\cdot)/\alpha, q(\cdot)/\alpha \in \mathbb{P}_1$. According to Proposition 2.2 and [14, Lemma 2.3] we have that

$$\begin{aligned}
I_1 &\leq C \left\| \sum_{j=\ell}^m \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}+\omega} \right\|_{p(\cdot),q(\cdot)} \\
&\leq C \left\| \sum_{j=\ell}^m (M_{HL}g_j)^{1/\alpha} \right\|_{p(\cdot),q(\cdot)} \\
&\leq C \left\| \left(\sum_{j=\ell}^m (M_{HL}g_j)^{1/\alpha} \right)^\alpha \right\|_{p(\cdot)/\alpha, q(\cdot)/\alpha}^{1/\alpha} \\
&\leq C \left\| \left(\sum_{j=\ell}^m g_j^{1/\alpha} \right)^\alpha \right\|_{p(\cdot)/\alpha, q(\cdot)/\alpha}^{1/\alpha} \\
(19) \quad &= C \left\| \sum_{j=\ell}^m \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}.
\end{aligned}$$

Suppose now that a is a $(p(\cdot), q(\cdot), \infty, s)$ -atom associated with $z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Let $m \in \mathbb{N}$. By proceeding as in [4, p. 19 and 20] we obtain

$$\begin{aligned}
M_\varphi(a)(x) &\leq C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} (b\lambda_-^{s+1})^{-m} \\
&\leq C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} b^{m(\gamma-1)} \lambda_-^{-m(s+1)} \left(\frac{1}{|B_{k+m+\omega+1}|} \int_{z+B_{k+m+\omega+1}} \chi_{z+B_k}(y) dy \right)^\gamma \\
&\leq C \frac{b^{m(\gamma-1)} \lambda_-^{-m(s+1)}}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} (M_{HL}(\chi_{z+B_k})(x))^\gamma, \quad x \in z + (B_{k+m+\omega+1} \setminus B_{k+m+\omega}).
\end{aligned}$$

Here γ is chosen such that $\gamma p(\cdot), \gamma q(\cdot) \in \mathbb{P}_1$. We now take $s \in \mathbb{N}$, satisfying that $b^{\gamma-1} \lambda_-^{-(s+1)} \leq 1$. We obtain

$$M_\varphi(a)(x) \leq C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} (M_{HL}(\chi_{z+B_k})(x))^\gamma, \quad x \notin z + B_{k+\omega}.$$

By proceeding as above we get

$$\begin{aligned}
I_2 &\leq C \left\| \sum_{j=\ell}^m \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \left(M_{HL}(\chi_{x_j+B_{\ell_j}}) \right)^\gamma \right\|_{p(\cdot),q(\cdot)} \\
&= C \left\| \left(\sum_{j=\ell}^m \left(\lambda_j^{1/\gamma} \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1/\gamma} M_{HL}(\chi_{x_j+B_{\ell_j}}) \right)^\gamma \right)^{1/\gamma} \right\|_{\gamma p(\cdot),\gamma q(\cdot)}^\gamma \\
(20) \quad &\leq C \left\| \sum_{j=\ell}^m \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}.
\end{aligned}$$

By combining (18), (19) and (20) we infer that the sequence $(\sum_{j=0}^k \lambda_j a_j)_{k \in \mathbb{N}}$ is Cauchy in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Since $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete (Proposition 2.5), the series $\sum_{j \in \mathbb{N}} \lambda_j a_j$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Moreover, we get

$$\left\| \sum_{j \in \mathbb{N}} \lambda_j a_j \right\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)} \leq C \left\| \sum_{j \in \mathbb{N}} \frac{\lambda_j}{\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}.$$

(ii) Assume that $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \cap L_{loc}^1(\mathbb{R}^n)$, $s \geq s_0$ (s_0 was defined in Proposition 3.2) and $N > \max\{N_0, s\}$ (N_0 was defined in Theorem 1.1). We recall that $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \cap L_{loc}^1(\mathbb{R}^n)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ (Corollary 3.1). Let $j \in \mathbb{Z}$. We define $\Omega_j = \{x \in \mathbb{R}^n : M_N(f)(x) > 2^j\}$. According to [4, Chapter 1, Section 5] we can write $f = g_j + \sum_{k \in \mathbb{N}} b_{j,k}$, that is, the Calderón-Zygmund decomposition of degree s and height 2^j associated with $M_N f$. The properties of g_j and $b_{j,k}$ will be specified when we need each of them.

As it was proved in Proposition 3.2, (ii), $g_j \rightarrow f$, as $j \rightarrow +\infty$, in both $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $S'(\mathbb{R}^n)$, and in Proposition 3.4 $g_j \rightarrow 0$, as $j \rightarrow -\infty$, in $S'(\mathbb{R}^n)$. We have that

$$f = \sum_{j \in \mathbb{Z}} (g_{j+1} - g_j), \quad \text{in } S'(\mathbb{R}^n).$$

As in [4, p. 38] we can write, for every $j \in \mathbb{Z}$,

$$g_{j+1} - g_j = \sum_{i \in \mathbb{N}} h_{i,j}, \quad \text{in } S'(\mathbb{R}^n),$$

where

$$h_{i,j} = (f - P_i^j) \zeta_i^j - \sum_{k \in \mathbb{N}} ((f - P_k^{j+1}) \zeta_i^j - P_{i,k}^{j+1}) \zeta_k^{j+1}, \quad i \in \mathbb{N}.$$

According to the properties of the polynomials P 's and the functions ζ 's it follows that, for every $P \in \mathcal{P}_s$,

$$\int_{\mathbb{R}^n} h_{i,j}(x) P(x) dx = 0, \quad i, j \in \mathbb{N}.$$

We also have that, for certain $C_0 > 0$, $\|h_{i,j}\|_\infty \leq C_0 2^j$ and $\text{supp } h_{i,j} \subset x_{i,j} + B_{\ell_{i,j}+4\omega}$, for every $i, j \in \mathbb{N}$ ([4, (6.12) and (6.13), p. 38]). Hence, for every $i, j \in \mathbb{N}$, the function $a_{i,j} = h_{i,j} 2^{-j} C_0^{-1} \|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|_{p(\cdot),q(\cdot)}^{-1}$ is a $(p(\cdot), q(\cdot), \infty, s)$ -atom. Moreover,

$$(21) \quad f = \sum_{i \in \mathbb{N}, j \in \mathbb{Z}} \lambda_{i,j} a_{i,j} \quad \text{in } S'(\mathbb{R}^n),$$

where $\lambda_{i,j} = 2^j C_0 \|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|_{p(\cdot),q(\cdot)}$, for every $i \in \mathbb{N}$, $j \in \mathbb{Z}$.

We are going to explain the convergence of the double series in (21).

We now choose $\beta > 1$ such that $\beta p, \beta q \in \mathbb{P}_1$. Assume that $\pi = (\pi_1, \pi_2) : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Z}$ is a bijection. By proceeding as before we get, for every $k \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{m=0}^k \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}} \right\|_{p(\cdot), q(\cdot)} &\leq C \left\| \sum_{m=0}^k 2^{\pi_2(m)} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}} \right\|_{p(\cdot), q(\cdot)} \\
&\leq C \left\| \sum_{m=0}^k \left(2^{\pi_2(m)/\beta} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}} \right)^\beta \right\|_{p(\cdot), q(\cdot)} \\
&\leq C \left\| \sum_{m=0}^k \left(2^{\pi_2(m)/\beta} M_{HL} \left(\chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 2\omega}} \right) \right)^\beta \right\|_{p(\cdot), q(\cdot)} \\
&= C \left\| \left(\sum_{m=0}^k \left(M_{HL} \left(2^{\pi_2(m)/\beta} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 2\omega}} \right) \right)^\beta \right)^{1/\beta} \right\|_{\beta p(\cdot), \beta q(\cdot)}^\beta \\
&\leq C \left\| \left(\sum_{m=0}^k 2^{\pi_2(m)} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 2\omega}} \right)^{1/\beta} \right\|_{\beta p(\cdot), \beta q(\cdot)}^\beta \\
&\leq C \left\| \sum_{j \in \mathbb{Z}} 2^j \sum_{i \in \mathbb{N}} \chi_{x_{i,j} + B_{\ell_{i,j} + 2\omega}} \right\|_{p(\cdot), q(\cdot)} \\
&\leq C \left\| \sum_{j \in \mathbb{Z}} 2^j \chi_{\Omega_j} \right\|.
\end{aligned}$$

Since $f \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$, by [22, Theorem 2.8 and Definition 2.5 vii)], $M_N(f)(x) < \infty$, a.e. $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ such that $M_N(f)(x) < \infty$. There exists $j_0 \in \mathbb{Z}$ such that $2^{j_0} < M_N(f)(x) \leq 2^{j_0+1}$. We have that

$$\sum_{j \in \mathbb{Z}} 2^j \chi_{\Omega_j}(x) = \sum_{j \leq j_0} 2^j = 2^{j_0+1} \leq 2M_N(f)(x).$$

We conclude that

$$\begin{aligned}
&\left\| \sum_{m=0}^k \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}} \right\|_{p(\cdot), q(\cdot)} \\
&= \left\| \left(\sum_{m=0}^k \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}} \right)^{1/\beta} \right\|_{\beta p(\cdot), \beta q(\cdot)}^\beta \leq C \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)},
\end{aligned}$$

where $C > 0$ does not depend on (k, π) .

According to [22, Theorem 2.8 and Definition 2.5, v)] we deduce that

$$\left\| \sum_{m \in \mathbb{N}} \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{\pi(m)} + B_{\ell_{\pi(m)} + 4\omega}} \right\|_{p(\cdot), q(\cdot)} \leq C \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}.$$

From the property we have just established in the part (i) of this proof we deduce that the series $\sum_{m \in \mathbb{N}} \lambda_{\pi(m)} a_{\pi(m)}$ converges both in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ and $S'(\mathbb{R}^n)$. Hence, for every $\phi \in S(\mathbb{R}^n)$, the series $\sum_{m \in \mathbb{N}} \lambda_{\pi(m)} \langle a_{\pi(m)}, \phi \rangle$ converges in \mathbb{C} .

Also we have that if $\Lambda : \mathbb{N} \times \mathbb{Z} \rightarrow \mathbb{N} \times \mathbb{Z}$ is a bijection, then the series $\sum_{m \in \mathbb{N}} \lambda_{\Lambda \circ \pi(m)} \langle a_{\Lambda \circ \pi(m)}, \phi \rangle$ converges in \mathbb{C} , for every $\phi \in S(\mathbb{R}^n)$. In other words, the series $\sum_{m \in \mathbb{N}} \lambda_{\pi(m)} \langle a_{\pi(m)}, \phi \rangle$ converges unconditionally in \mathbb{C} , for every $\phi \in S(\mathbb{R}^n)$. Hence, $\sum_{m \in \mathbb{N}} \lambda_{\pi(m)} \langle a_{\pi(m)}, \phi \rangle < \infty$, for every $\phi \in S(\mathbb{R}^n)$.

Let $\phi \in S(\mathbb{R}^n)$. Since $\sum_{m \in \mathbb{N}} \lambda_{\pi(m)} \langle a_{\pi(m)}, \phi \rangle < \infty$, the double series $\sum_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \lambda_{i,j} \langle a_{i,j}, \phi \rangle$ is summable, that is, $\sup_{m \in \mathbb{N}} \sum_{1 \leq i \leq m, |j| \leq m} \lambda_{i,j} \langle a_{i,j}, \phi \rangle < \infty$. Then, for every bijection $\pi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Z}$, we have that

$$\langle f, \phi \rangle = \sum_{i \in \mathbb{N}} \left(\sum_{j \in \mathbb{Z}} \lambda_{i,j} \langle a_{i,j}, \phi \rangle \right) = \sum_{m \in \mathbb{N}} \lambda_{\pi(m)} \langle a_{\pi(m)}, \phi \rangle.$$

Suppose now that $f \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. Then, there exists a sequence $\{f_j\}_{j \in \mathbb{N}}$ in $L^1_{loc}(\mathbb{R}^n) \cap H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ such that $f_1 = 0$, $f_j \rightarrow f$, as $j \rightarrow \infty$, in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$, and $\|f_{j+1} - f_j\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)} < 2^{-j} \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}$, for every $j \in \mathbb{N}$. Then, we can write

$$f = \sum_{j \in \mathbb{N}} (f_{j+1} - f_j),$$

in the sense of convergence in both $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$ and $S'(\mathbb{R}^n)$. For every $j \in \mathbb{N}$, there exist a sequence $\{\lambda_{i,j}\}_{i \in \mathbb{N}} \subset (0, \infty)$ and a sequence $\{a_{i,j}\}_{i \in \mathbb{N}}$ of $(p(\cdot), q(\cdot), \infty, s)$ -atoms, being for every $i \in \mathbb{N}$, $a_{i,j}$ associated with $x_{i,j} \in \mathbb{R}^n$ and $\ell_{i,j} \in \mathbb{Z}$, satisfying that

$$f_{j+1} - f_j = \sum_{i \in \mathbb{N}} \lambda_{i,j} a_{i,j}, \quad \text{in } S'(\mathbb{R}^n),$$

and

$$\left\| \sum_{i \in \mathbb{N}} \frac{\lambda_{i,j}}{\|\chi_{x_{i,j} + B_{\ell_{i,j}}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{i,j} + B_{\ell_{i,j}}} \right\|_{p(\cdot), q(\cdot)} \leq C 2^{-j} \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}.$$

Here $C > 0$ does not depend on f .

We have that

$$\begin{aligned} \left\| \sum_{i \in \mathbb{N}, j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\|\chi_{x_{i,j} + B_{\ell_{i,j}}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{i,j} + B_{\ell_{i,j}}} \right\|_{p(\cdot), q(\cdot)} &\leq \sum_{j \in \mathbb{Z}} \left\| \sum_{i \in \mathbb{N}} \frac{\lambda_{i,j}}{\|\chi_{x_{i,j} + B_{\ell_{i,j}}}\|_{p(\cdot), q(\cdot)}} \chi_{x_{i,j} + B_{\ell_{i,j}}} \right\|_{p(\cdot), q(\cdot)} \\ &\leq C \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)}. \end{aligned}$$

By proceeding as above we can write

$$f = \sum_{m \in \mathbb{N}} \lambda_{\pi(m)} a_{\pi(m)}, \quad \text{in } S'(\mathbb{R}^n),$$

for every bijection $\pi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$.

Thus the proof of this case is finished.

4.2. Proof of Theorem 1.2 when $r < \infty$. In order to prove this property we proceed in a series of steps establishing auxiliary and partial results.

Proposition 4.1. *Let $1 < r < \infty$ and let $p, q \in \mathbb{P}_0$. There exists $s_0 \in \mathbb{N}$ satisfying that if $s \in \mathbb{N}$, $s \geq s_0$, we can find $C > 0$ for which, for every $f \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$ and a $(p(\cdot), q(\cdot), r, s)$ -atom a_j associated with some $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, such that*

$$\left\| \sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j + B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j + B_{\ell_j}} \right\|_{p(\cdot), q(\cdot)} \leq C \|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)},$$

and $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$.

Proof. Suppose that a is a $(p(\cdot), q(\cdot), \infty, s)$ -atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We have that

$$\|a\|_r = \left(\int_{x_0 + B_k} |a(x)|^r dx \right)^{1/r} \leq b^{k/r} \|a\|_\infty \leq b^{k/r} \|\chi_{x_0 + B_k}\|_{p(\cdot), q(\cdot)}^{-1}.$$

Hence, a is a $(p(\cdot), q(\cdot), r, s)$ -atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Then, this property follows from the previous case $r = \infty$. \square

We are going to see that the $(p(\cdot), q(\cdot), r, s)$ -atoms are in $H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.

Proposition 4.2. *Let $p, q \in \mathbb{P}_0$ such that $p(0) \leq q(0)$. Assume that $\max\{1, q_+\} < r < \infty$. There exists $s_0 \in \mathbb{N}$ such that if a is a $(p(\cdot), q(\cdot), r, s_0)$ -atom, then $a \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.*

Proof. Let $\varphi \in S(\mathbb{R}^n)$. Assume that a is a $(p(\cdot), q(\cdot), r, s)$ -atom associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$, where $s \in \mathbb{N}$ will be specified later. We have that

$$\|M_\varphi(a)\|_{p(\cdot), q(\cdot)} \leq C \left(\|M_\varphi(a) \chi_{x_0 + B_{\ell_0 + w}}\|_{p(\cdot), q(\cdot)} + \|M_\varphi(a) \chi_{(x_0 + B_{\ell_0 + w})^c}\|_{p(\cdot), q(\cdot)} \right) = I_1 + I_2.$$

It is clear that

$$(M_\varphi(a) \chi_{x_0 + B_{\ell_0 + w}})^*(t) = 0, \quad t \geq |x_0 + B_{\ell_0 + w}| = b^{\ell_0 + w}.$$

Then, since $q(0) \geq p(0)$, we can write

$$I_1 \leq C \left\| t^{1/p(t)-1/q(t)} (M_\varphi(a))^* \chi_{(0, b\ell_0+w)} \right\|_{q(\cdot)} \leq C \left\| (M_\varphi(a))^* \chi_{(0, b\ell_0+w)} \right\|_{q(\cdot)}.$$

By using [14, Lemma 2.2] and since $r > \max\{1, q_+\}$ we obtain

$$I_1 \leq C \|(M_\varphi(a))^*\|_{L^r(0, \infty)} = C \|M_\varphi(a)\|_{L^r(\mathbb{R}^n)} \leq C \|a\|_{L^r(\mathbb{R}^n)} \leq C b^{\ell_0/r} \|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot), q(\cdot)}^{-1} < \infty.$$

By proceeding as in the proof of the case $r = \infty$ (see [4, p. 19-21]) we get

$$M_\varphi(a)(x) \leq \frac{C}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot), q(\cdot)}} (M_{HL}(\chi_{x_0+B_{\ell_0}})(x))^\gamma, \quad x \notin x_0 + B_{\ell_0+w},$$

provided that $s \geq \frac{\gamma-1}{\log_b(\lambda_-)} - 1$, where $\gamma > 1$ is such that $\gamma p, \gamma q \in \mathbb{P}_1$. Then, Proposition 2.1 implies that

$$I_2 \leq C \frac{\|(M_{HL}(\chi_{x_0+B_{\ell_0}}))^\gamma\|_{p(\cdot), q(\cdot)}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot), q(\cdot)}} = C \frac{\|M_{HL}(\chi_{x_0+B_{\ell_0}})\|_{\gamma p(\cdot), \gamma q(\cdot)}^\gamma}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot), q(\cdot)}} \leq C.$$

Thus, we have shown that $a \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$. \square

Note that the constant C in the proof of the last proposition depends on the atom a . This fact says us that we can not prove next result as a consequence of Proposition 4.2. We need to make a more involved work to show the following property.

Proposition 4.3. *Let $p, q \in \mathbb{P}_0$ being $p(0) \leq q(0)$. There exist $s_0 \in \mathbb{N}$ and $r_0 > 1$ such that, for every $r \geq r_0$ we can find $C > 0$ satisfying that if, for every $j \in \mathbb{N}$, $\lambda_j > 0$ and a_j is a $(p(\cdot), q(\cdot), r, s_0)$ -atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$ such that*

$$\sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot), q(\cdot)}(\mathbb{R}^n),$$

then $f = \sum_{j \in \mathbb{N}} \lambda_j a_j \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$, and

$$\|f\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)} \leq C \left\| \sum_{j \in \mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot), q(\cdot)}.$$

In order to prove this proposition we need to establish some previous properties.

Lemma 4.1. *Assume that $(\lambda_k)_{k \in \mathbb{N}}$ is a sequence in $(0, \infty)$, $(\ell_k)_{k \in \mathbb{N}}$ is a sequence in \mathbb{Z} , $(x_k)_{k \in \mathbb{N}}$ is a sequence in \mathbb{R}^n , ν is a doubling weight (with respect to the anisotropic balls), $\ell \in \mathbb{N}$, $\ell \geq 1$, and $0 < p < \infty$. Then,*

$$(22) \quad \left\| \sum_{k \in \mathbb{N}} \lambda_k \chi_{x_k+B_{\ell_k+\ell}} \right\|_{L^p(\mathbb{R}^n, \nu)} \leq C b^{\ell\delta} \left\| \sum_{k \in \mathbb{N}} \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)}.$$

Here $C, \delta > 0$ depends only on ν .

Proof. Suppose firstly that $p > 1$. We follow the ideas in the proof of [45, Theorem 2, p. 53]. We take $0 \leq g \in L^{p'}(\mathbb{R}^n, \nu)$, where p' is the exponent conjugated to p , that is, $p' = p/(p-1)$. Let $y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We define the maximal operator M_ν by

$$M_\nu(h)(z) = \sup_{m \in \mathbb{Z}, y \in z+B_m} \frac{1}{\nu(y+B_m)} \int_{y+B_m} |h(x)| \nu(x) dx, \quad z \in \mathbb{R}^n.$$

Since ν is doubling with respect to the anisotropic balls, for a certain $\delta > 0$, we have that

$$\begin{aligned} \int_{y+B_{k+\ell}} g(x) \nu(x) dx &\leq b^{\ell\delta} \frac{\nu(y+B_k)}{\nu(y+B_{k+\ell})} \int_{y+B_{k+\ell}} g(x) \nu(x) dx \\ &\leq b^{\ell\delta} \int_{y+B_k} M_\nu(g)(x) \nu(x) dx, \quad y \in \mathbb{R}^n, k \in \mathbb{Z}. \end{aligned}$$

We have taken into account that

$$M_\nu(g)(z) \geq \frac{1}{\nu(y+B_{k+\ell})} \int_{y+B_{k+\ell}} g(x) \nu(x) dx, \quad z \in y+B_k.$$

Let $m \in \mathbb{N}$. We can write

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}}(x) g(x) \nu(x) dx &= \sum_{k=0}^m \lambda_k \int_{x_k+B_{\ell_k}} g(x) \nu(x) dx \\ &\leq b^{\ell\delta} \sum_{k=0}^m \lambda_k \int_{x_k+B_{\ell_k}} M_\nu(g)(x) \nu(x) dx. \end{aligned}$$

Hence, the maximal theorem [45, Theorem 3, p. 3] leads to

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}}(x) g(x) \nu(x) dx \right| &\leq b^{\ell\delta} \left\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)} \|M_\nu(g)\|_{L^{p'}(\mathbb{R}^n, \nu)} \\ &\leq C b^{\ell\delta} \left\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)} \|g\|_{L^{p'}(\mathbb{R}^n, \nu)}. \end{aligned}$$

We conclude that

$$\left\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}}(x) \right\|_{L^p(\mathbb{R}^n, \nu)} \leq C b^{\ell\delta} \left\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)}.$$

By taking $m \rightarrow \infty$, the monotone convergence theorem allows us to establish (22) in this case.

Assume now that $0 < p \leq 1$. For every $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$, we denote by $\delta_{(x_0, k_0)}$ the Dirac measure in \mathbb{R}^{n+1} supported in (x_0, k_0) . Let $m \in \mathbb{N}$. We have that

$$\begin{aligned} \int_{x \in y+B_{\ell_j}} \sum_{k=0}^m \lambda_k \delta_{(x_k, \ell_k)}(y, j) &= \sum_{k=0}^m \lambda_k \int_{\mathbb{R}^{n+1}} \chi_{\{(y, j): x \in y+B_{\ell_j}\}}(y, j) \delta_{(x_k, \ell_k)}(y, j) \\ &= \sum_{k=0}^m \lambda_k \chi_{\{(y, j): x \in y+B_{\ell_j}\}}(x_k, \ell_k) = \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}}(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

Also, we can write

$$\int_{x \in y+B_j} \sum_{k=0}^m \lambda_k \delta_{(x_k, \ell_k)}(y, j) = \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}}(x), \quad x \in \mathbb{R}^n.$$

By arguing as in the proof of [45, Theorem 1, p. 52] replacing the area Littlewood-Paley functions by our area integrals we can prove that

$$\left\| \int_{x \in y+B_{\ell_j}} \sum_{k=0}^m \lambda_k \delta_{(x_k, \ell_k)}(y, j) \right\|_{L^p(\mathbb{R}^n, \nu)} \leq C b^{\ell\delta} \left\| \int_{x \in y+B_j} \sum_{k=0}^m \lambda_k \delta_{(x_k, \ell_k)}(y, j) \right\|_{L^p(\mathbb{R}^n, \nu)}.$$

By letting $m \rightarrow \infty$ and using monotone convergence theorem we conclude (22). \square

We now recall definitions of anisotropic \mathcal{A}_r -weights and anisotropic weighted Hardy spaces (see [5] and [45]).

Let $r \in (1, \infty)$ and ν be a nonnegative measurable function on \mathbb{R}^n . The function ν is said to be a weight in the anisotropic Muckenhoupt class $\mathcal{A}_r(\mathbb{R}^n, A)$ when

$$[\nu]_{\mathcal{A}_r(\mathbb{R}^n, A)} =: \sup_{x \in \mathbb{R}^n, k \in \mathbb{Z}} \left(\frac{1}{|B_k|} \int_{x+B_k} \nu(y) dy \right) \left(\frac{1}{|B_k|} \int_{x+B_k} (\nu(y))^{-1/(r-1)} dy \right)^{r-1} < \infty.$$

We say that ν belongs to the anisotropic Muckenhoupt class $\mathcal{A}_1(\mathbb{R}^n, A)$ when

$$[\nu]_{\mathcal{A}_1(\mathbb{R}^n, A)} =: \sup_{x \in \mathbb{R}^n, k \in \mathbb{Z}} \left(\frac{1}{|B_k|} \int_{x+B_k} \nu(y) dy \right) \sup_{y \in x+B_k} (\nu(y))^{-1} < \infty.$$

We define $\mathcal{A}_\infty(\mathbb{R}^n, A) = \bigcup_{1 \leq r < \infty} \mathcal{A}_r(\mathbb{R}^n, A)$.

The weight ν satisfies the reverse Hölder condition $RH_r(\mathbb{R}^n, A)$ (in short, $\nu \in RH_r(\mathbb{R}^n, A)$) if there exists $C > 0$ such that

$$\left(\frac{1}{|B_k|} \int_{x+B_k} (\nu(y))^r dy \right)^{1/r} \leq C \frac{1}{|B_k|} \int_{x+B_k} \nu(y) dy, \quad x \in \mathbb{R}^n \text{ and } k \in \mathbb{Z}.$$

The classes $\mathcal{A}_r(\mathbb{R}^n, A)$ and $RH_\alpha(\mathbb{R}^n, A)$ are closely connected. In particular, if $\nu \in \mathcal{A}_1(\mathbb{R}^n, A)$, there exists $\alpha \in (1, \infty)$ such that $\nu \in RH_\alpha(\mathbb{R}^n, A)$ ([33, Theorem 1.3]).

Let $1 \leq r < \infty$ and $\nu \in \mathcal{A}_r(\mathbb{R}^n, A)$. For every $N \in \mathbb{N}$, the anisotropic Hardy space $H_N^r(\mathbb{R}^n, \nu, A)$ consists of all those $f \in S'(\mathbb{R}^n)$ such that $M_N(f) \in L^r(\mathbb{R}^n, \nu)$. There exists $N_{r,\nu} \in \mathbb{N}$ satisfying that $H_N^r(\mathbb{R}^n, \nu, A) = H_{N_{r,\nu}}^r(\mathbb{R}^n, \nu, A)$, for every $N \geq N_{r,\nu}$. Moreover, when $N \geq N_{r,\nu}$ the quantities $\|M_N(f)\|_{L^r(\mathbb{R}^n, \nu)}$ and $\|M_{N_{r,\nu}}(f)\|_{L^r(\mathbb{R}^n, \nu)}$ are equivalent, for every $f \in H_{N_{r,\nu}}^r(\mathbb{R}^n, \nu, A)$. We denote $H^r(\mathbb{R}^n, \nu, A)$ to the space $H_{N_{r,\nu}}^r(\mathbb{R}^n, \nu, A)$.

By proceeding as in the proof of [45, Lemma 5, p. 116] we can obtain the following property.

Lemma 4.2. *Let $p \in (0, \infty)$ and $q > \max\{1, p\}$. Assume that $\nu \in RH_{(q/p)'}(\mathbb{R}^n, A)$. Then, there exists $C > 0$ such that if, for every $k \in \mathbb{N}$, the measurable function a_k has its support contained in the ball $x_k + B_{\ell_k}$, where $x_k \in \mathbb{R}^n$, $\ell_k \in \mathbb{Z}$, $\|a_k\|_q \leq \|\chi_{x_k + B_{\ell_k}}\|_q$, and $\lambda_k > 0$, we have that*

$$\left\| \sum_{k \in \mathbb{N}} \lambda_k a_k \right\|_{L^p(\mathbb{R}^n, \nu)} \leq C \left\| \sum_{k \in \mathbb{N}} \lambda_k \chi_{x_k + B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)}.$$

If $1 < r \leq \infty$ and $N \in \mathbb{N}$ we say that a function $a \in L^r(\mathbb{R}^n)$ is a (r, N) -atom associated with $x_0 \in \mathbb{R}^n$ and $j_0 \in \mathbb{Z}$, when a satisfies the following properties:

- (i) $\text{supp } a \subset x_0 + B_{j_0}$,
- (ii) $\|a\|_r \leq b^{j_0/r}$,
- (iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$, for all $s(\alpha) \leq N$, $\alpha \in \mathbb{N}^n$.

Next result is an anisotropic version of the second part of [45, Theorem 1, p. 112].

Lemma 4.3. *Let $0 < p < \infty$. Assume that $\nu \in RH_{(q/p)'}(\mathbb{R}^n, A)$ where $q > \max\{1, p\}$. There exists $N_1 \in \mathbb{N}$ and $C > 0$ such that if, for every $k \in \mathbb{N}$, a_k is a (q, N_1) -atom associated with $x_k \in \mathbb{R}^n$ and $\ell_k \in \mathbb{Z}$, and $\lambda_k > 0$, satisfying that*

$$\left\| \sum_{k=1}^{\infty} \lambda_k \chi_{x_k + B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)} < \infty,$$

then the series $\sum_{k=1}^{\infty} \lambda_k a_k$ converges in both $S'(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n, \nu, A)$ to an element $f \in H^p(\mathbb{R}^n, \nu, A)$ such that

$$\|f\|_{H^p(\mathbb{R}^n, \nu, A)} \leq C \left\| \sum_{k=1}^{\infty} \lambda_k \chi_{x_k + B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n, \nu)}.$$

Proof. Suppose that a is a (q, N) -atom associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$. Here $N \in \mathbb{N}$ will be specified later.

We choose $\varphi \in S(\mathbb{R}^n)$. We now estimate $\|M_\varphi(a)\|_{L^q(\mathbb{R}^n)}$ by considering in a separate way the regions $x_0 + B_{\ell_0+w}$ and $(x_0 + B_{\ell_0+w})^c$.

Since $q > 1$ the maximal theorem ([4, Theorem 3.6]) implies that

$$\left(\int_{\mathbb{R}^n} \chi_{x_0 + B_{\ell_0+w}}(x) |M_\varphi(a)(x)|^q dx \right)^{1/q} \leq \|M_\varphi(a)\|_{L^q(\mathbb{R}^n)} \leq C b^{\ell_0/q} \leq C b^{(\ell_0+w)/q}.$$

Hence, the function $\beta_0 = \frac{1}{C} \chi_{x_0 + B_{\ell_0+w}} M_\varphi(a)$ is a $(q, -1)$ -atom associated with x_0 and $\ell_0 + w$. The index -1 means that no null moment condition need to be satisfied.

By proceeding as in [4, p. 20] we get, for every $m \in \mathbb{N}$,

$$M_\varphi(a)(x) \leq C (b \lambda_-^{N+1})^{-m}, \quad x \in x_0 + (B_{\ell_0+w+m+1} \setminus B_{\ell_0+w+m}).$$

We define $\rho_m = \chi_{x_0 + B_{\ell_0+w+m+1}}$, $m \in \mathbb{N}$. It is clear that ρ_m is a $(q, -1)$ -atom associated with x_0 and $\ell_0 + w + m + 1$, for every $m \in \mathbb{N}$, and that

$$\chi_{(x_0 + B_{\ell_0+w})^c} M_\varphi(a) \leq C \sum_{m \in \mathbb{N}} (b \lambda_-^{N+1})^{-m} \rho_m.$$

Hence, we obtain

$$(23) \quad M_\varphi(a) \leq C \left(\beta_0 + \sum_{m \in \mathbb{N}} (b \lambda_-^{N+1})^{-m} \rho_m \right).$$

Here $C > 0$ does not depend on a .

Suppose that $k \in \mathbb{N}$ and, for every $j \in \mathbb{N}$, $j \leq k$, $\lambda_j > 0$ and a_j is a (q, N) -atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$. According to (23) we get

$$M_\varphi \left(\sum_{j=0}^k \lambda_j a_j \right) \leq C \left(\sum_{j=0}^k \lambda_j (\beta_{0,j} + \sum_{m=0}^{\infty} (b\lambda_-^{N+1})^{-m} \rho_{m,j}) \right),$$

where $\beta_{0,j}$ and $\rho_{m,j}$, $j = 1, \dots, k$, and $m \in \mathbb{N}$ have the obvious meaning and are $(q, -1)$ -atoms. By using Lemmas 4.1 and 4.2, and by taking $p_1 = \min\{1, p\}$ we have that

$$\begin{aligned} \left\| \sum_{j=0}^k \lambda_j a_j \right\|_{H^p(\mathbb{R}^n, \nu, A)}^{p_1} &\leq C \left\| \sum_{j=0}^k \lambda_j (\beta_{0,j} + \sum_{m \in \mathbb{N}} (b\lambda_-^{N+1})^{-m} \rho_{m,j}) \right\|_{L^p(\mathbb{R}^n, \nu)}^{p_1} \\ &\leq C \left(\sum_{m \in \mathbb{N}} (b\lambda_-^{N+1})^{-mp_1} \left\| \sum_{j=0}^k \lambda_j \rho_{m,j} \right\|_{L^p(\mathbb{R}^n, \nu)}^{p_1} + \left\| \sum_{j=0}^k \lambda_j \beta_{0,j} \right\|_{L^p(\mathbb{R}^n, \nu)}^{p_1} \right) \\ &\leq C \left(\sum_{m \in \mathbb{N}} (b\lambda_-^{N+1})^{-mp_1} \left\| \sum_{j=0}^k \lambda_j \chi_{x_j + B_{\ell_j + w + m + 1}} \right\|_{L^p(\mathbb{R}^n, \nu)}^{p_1} + \left\| \sum_{j=0}^k \lambda_j \chi_{x_j + B_{\ell_j}} \right\|_{L^p(\mathbb{R}^n, \nu)}^{p_1} \right) \\ &\leq C \left(\sum_{m \in \mathbb{N}} (b\lambda_-^{N+1})^{-mp_1} b^{\delta mp_1} + 1 \right) \left\| \sum_{j=0}^k \lambda_j \chi_{x_j + B_{\ell_j + w}} \right\|_{L^p(\mathbb{R}^n, \nu)}^{p_1}, \end{aligned}$$

for a certain $\delta > 0$. Hence, if $(\delta - 1) \ln b / \ln(\lambda_-) < N + 1$, we conclude that

$$\left\| \sum_{j=0}^k \lambda_j a_j \right\|_{H^p(\mathbb{R}^n, \nu, A)} \leq C \left\| \sum_{j=0}^k \lambda_j \chi_{x_j + B_{\ell_j}} \right\|_{L^p(\nu)}.$$

Standard arguments allow us to finish the proof of this property. \square

From Lemma 4.3 we can deduce the following.

Lemma 4.4. Assume that $p, q \in \mathbb{P}_0$, $p_0 \in (0, \infty)$, $q_0 > \max\{1, p_0\}$ and $\nu \in \mathcal{A}_1(\mathbb{R}^n, A) \cap RH_{(q_0/p_0)'}(\mathbb{R}^n, A)$. Suppose that, for every $k \in \mathbb{N}$, $\lambda_k > 0$ and a_k is a $(p(\cdot), q(\cdot), q_0, N_1)$ -atom associated with $x_k \in \mathbb{R}^n$ and $\ell_k \in \mathbb{Z}$, satisfying that

$$\left\| \sum_{k \in \mathbb{N}} \lambda_k \left\| \chi_{x_k + B_{\ell_k}} \right\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_k + B_{\ell_k}} \right\|_{L^{p_0}(\mathbb{R}^n, \nu)} < \infty.$$

Here N_1 is the one defined in Lemma 4.3.

Then, the series $f = \sum_{k \in \mathbb{N}} \lambda_k a_k$ converges in $H^{p_0}(\mathbb{R}^n, \nu, A)$ and

$$\|f\|_{H^{p_0}(\mathbb{R}^n, \nu, A)} \leq C \left\| \sum_{k \in \mathbb{N}} \lambda_k \left\| \chi_{x_k + B_{\ell_k}} \right\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_k + B_{\ell_k}} \right\|_{L^{p_0}(\mathbb{R}^n, \nu)}.$$

Here C does not depend on $\{\lambda_k\}_{k \in \mathbb{N}}$ and $\{a_k\}_{k \in \mathbb{N}}$.

Proof. It is sufficient to note that, for every $k \in \mathbb{N}$, $a_k \left\| \chi_{x_k + B_{\ell_k}} \right\|_{p(\cdot), q(\cdot)}$ is a (q_0, N_1) -atom and ν is doubling with respect to anisotropic balls. \square

Proof of Proposition 4.3. We choose $\alpha > 1$ such that $\alpha p, \alpha q \in \mathbb{P}_1$. Then, $(\alpha p)', (\alpha q)' \in \mathbb{P}_1$. We recall that the dual space $(\mathcal{L}^{\alpha p(\cdot), \alpha q(\cdot)}(\mathbb{R}^n))^*$ of $\mathcal{L}^{\alpha p(\cdot), \alpha q(\cdot)}(\mathbb{R}^n)$ is $\mathcal{L}^{(\alpha p(\cdot))', (\alpha q(\cdot))'}(\mathbb{R}^n)$ and the maximal operator M_{HL} is bounded from $\mathcal{L}^{(\alpha p(\cdot))', (\alpha q(\cdot))'}(\mathbb{R}^n)$ into itself (Proposition 2.1).

In the sequel our argument is as in [14] supported in the Rubio de Francia iteration algorithm. Given a function h we define $M_{HL}^0(h) = |h|$ and, for every $i \in \mathbb{N}$, $i \geq 1$, $M_{HL}^i(h) = M_{HL} \circ M_{HL}^{i-1}(h)$. We consider

$$R(h) = \sum_{i=0}^{\infty} \frac{M_{HL}^i(h)}{2^i \|M_{HL}\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'}^i}.$$

We have that

- (i) $|h| \leq R(h)$;
- (ii) R is bounded from $\mathcal{L}^{(\alpha p(\cdot))', (\alpha q(\cdot))'}(\mathbb{R}^n)$ into itself and $\|R(h)\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'} \leq 2\|h\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'}$;
- (iii) $R(h) \in \mathcal{A}_1(\mathbb{R}^n, A)$ and $[R(h)]_{\mathcal{A}_1(\mathbb{R}^n, A)} \leq 2\|M_{HL}\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'}$. Hence, there exists $\beta_0 > 1$ such that $R(h) \in RH_{\beta_0}(\mathbb{R}^n, A)$.

We choose $r > \max\{1, q_+\}$ such that $R(h) \in RH_{(r\alpha)'}(\mathbb{R}^n, A)$. It is sufficient to take $r > \max\{1, q_+, \frac{\beta_0}{\alpha(\beta_0-1)}\}$.

Suppose that $k \in \mathbb{N}$ and, for every $j \in \mathbb{N}$, $j \leq k$, $\lambda_j > 0$ and a_j is a $(p(\cdot), q(\cdot), r, N_1)$ -atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$. Here N_1 is the one defined in Lemma 4.3. We define $f_k = \sum_{j=0}^k \lambda_j a_j$. According to Proposition 4.2, $f_k \in H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)$.

Since $R(h) \in \mathcal{A}_1(\mathbb{R}^n, A) \cap RH_{(r\alpha)'}(\mathbb{R}^n, A)$, by Lemma 4.4, $f_k \in H^{1/\alpha}(\mathbb{R}^n, R(h), A)$ and

$$(24) \quad \|f_k\|_{H^{1/\alpha}(\mathbb{R}^n, R(h), A)} \leq C \left\| \sum_{j=0}^k \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{L^{1/\alpha}(\mathbb{R}^n, R(h))}.$$

Let $\varphi \in S(\mathbb{R}^n)$. By using [14, Lemma 2.3] and [22, Lemma 2.7] we can write

$$\|M_\varphi(f_k)\|_{p(\cdot), q(\cdot)}^{1/\alpha} = \|(M_\varphi(f_k))^{1/\alpha}\|_{\alpha p(\cdot), \alpha q(\cdot)} \leq C \sup_h \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} h(x) dx,$$

where the supremum is taken over all the functions $0 \leq h \in \mathcal{L}^{(\alpha p(\cdot))', (\alpha q(\cdot))'}(\mathbb{R}^n)$ such that $\|h\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'} \leq 1$.

By the above properties (i), (ii) and (iii) and (24), for every $0 \leq h \in \mathcal{L}^{(\alpha p(\cdot))', (\alpha q(\cdot))'}(\mathbb{R}^n)$ such that $\|h\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'} \leq 1$, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} h(x) dx &\leq \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} R(h)(x) dx \\ &\leq C \int_{\mathbb{R}^n} \left(\sum_{j=0}^k \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}}(x) \right)^{1/\alpha} R(h)(x) dx \\ &\leq C \left\| \left(\sum_{j=0}^k \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right)^{1/\alpha} \right\|_{\alpha p(\cdot), \alpha q(\cdot)} \|R(h)\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'} \\ &\leq C \left\| \sum_{j=0}^k \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot), q(\cdot)}^{1/\alpha} \|h\|_{(\alpha p(\cdot))', (\alpha q(\cdot))'} \\ &\leq C \left\| \sum_{j=0}^k \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot), q(\cdot)}^{1/\alpha}. \end{aligned}$$

Hence, we obtain

$$\|f_k\|_{H^{p(\cdot), q(\cdot)}(\mathbb{R}^n, A)} \leq C \left\| \sum_{j=0}^k \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot), q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot), q(\cdot)}^{1/\alpha}.$$

We finish the proof by using standard arguments. \square

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